Maximal and minimal polyiamonds

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Abstract

The minimum perimeter of an n-polyiamond is whichever of $\lceil\sqrt{6n}\rceil$ or $\lceil\sqrt{6n}\rceil + 1$ has the same parity as n. To prove this result, we first obtain a lower bound on the perimeter by considering maximal polyiamonds (i.e., polyiamonds with a given perimeter and a maximum number of triangles). We then show how to construct minimal polyiamonds that attain the perimeter lower bounds.

The maximum number of triangles in a polyiamond with perimeter p is round $(p^2/6) - \delta_6$, where δ_6 is 0 if $p \equiv 0 \pmod 6$, and is 1 else.

Keywords: polyiamond, perimeter, maximal, minimal

1 Introduction

A *polyiamond* is a connected planar set of congruent equilateral triangles in which the edges of adjacent triangles line up exactly (are not staggered).

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"Polyiamond" is a generalization of "diamond". An *n-polyiamond* is a polyiamond with n triangles. See Figure 1. We assume here that each edge of each triangle has length 1 and define the *perimeter* of a polyiamond to be the total length of its exposed edges.

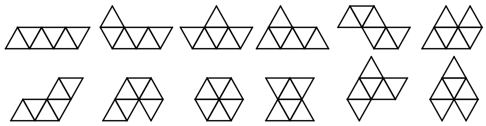

Figure 1: The 12 polyiamonds with 6 triangles. All have perimeter 8 except for the regular hexagon, which has perimeter 6.

A polyiamond is *minimal* (or *optimal*) iff it has min perimeter with respect to all polyiamonds with the same number of triangles. (The use of the term "optimal" comes from the minimum perimeter problem with polyominos, discussed below.) A polyiamond is *maximal* iff it has the maximum number of triangles with respect to all polyiamonds of the same perimeter. Minimal polyiamonds are useful in the solution to domain decomposition problems in scientific computation (see [7] for an illustration of the use of domain decomposition in conjunction with triangulations).

Note that there are 2 other possibilities: we could maximize the perimeter subject to a fixed number of triangles, or minimize the number of triangles subject to a fixed perimeter. But these problems are trivial and have the same solution: the polyiamond shaped like a stick.

The main results of this paper are:

- **Maximal polyiamond formula**: The maximum number of triangles in a polyiamond with perimeter p is

$$\text{round}\left(\frac{p^2}{6}\right) - \begin{cases} 0 & \text{else if } p \equiv 0 \pmod 6 \\ 1 & \text{else} \end{cases}$$

- **Maximal polyiamond algorithm**: Construct a polyiamond with perimeter p and the most triangles. (These polyiamonds are hexagons, except for small values of p.)

- **Minimal polyiamond formula**: The min perimeter of an n-polyiamond is whichever of $\lceil\sqrt{6n}\rceil$ or $\lceil\sqrt{6n}\rceil + 1$ has the same parity as n.

- **Minimal polyiamond algorithm**: Construct an n-polyiamond with min perimeter. (These polyiamonds are hexagons or “near-hexagons”.)

These results are related as follows: a lower bound on perimeter is obtained using maximal polyiamonds, and then attainment of the lower bound is demonstrated by using the minimal polyiamond algorithm.

Motivation for this paper came from the domain decomposition problem with polyominos, which is a whole topic by itself; see the references for details. The many approaches to this problem include branch-and-bound, genetic algorithms, knapsack algorithms, and stripe algorithms.

Below, we briefly discuss this domain decomposition problem and some known results. Although the results are well-known, we present them in terms of polyominos, and these results for polyominos will motivate the four main results for polyiamonds in the following sections.

2 Domain decomposition problem with polyominos

A *polyomino* is a connected planar set of congruent squares in which the edges of adjacent squares line up exactly (are not staggered). Equivalently, if we move a rook on a chessboard of any finite size, then the set of squares touched by the rook is a polyomino. “Polyomino” is a generalization of “domino”. An *n-polyomino* is a polyomino of n squares. See Figure 2.

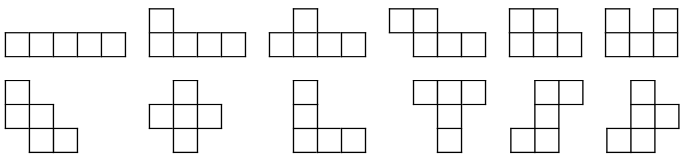

Figure 2: The 12 polyominos with 5 squares. All have perimeter 12 except for one shaped like a 1×1 square joined to a 2×2 square, which has perimeter 10.

The *domain decomposition problem* is a special case of graph partitioning problems which involve partitioning the vertices of a graph into equal-size sets as to minimize the number of edges connecting vertices in different sets. One version of the domain decomposition problem is as follows.

Problem 1. (**Domain decomposition problem with polyominos**) *Let n divide A. Tile a given set of A squares with n-polyominos. What is the min total perimeter of the polyominos in such a tiling?*

This paper arose when we asked what would happen if we worked with equilateral triangles instead of squares. The domain decomposition problem with polyominos has motivation from parallel computation; think of the following analogy:

square	job that needs to communicate with adjacent jobs
n-polyomino	n jobs assigned to a processor
polyomino edge	expensive communication between jobs in different processors

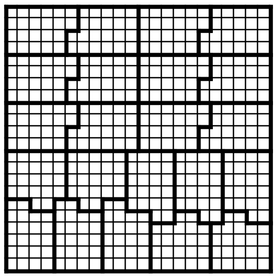

Figure 3: A solution of the domain decomposition problem with polyominos for a 22×22 board tiled by 22 polyominos, each of which has 22 squares and has perimeter 20. By Yackel-Meyer-Christou's theorem, the min perimeter of a 22-polyomino is 20. So the total perimeter lower bound is $22 \times 20 = 440$ and is attained by this tiling.

In the domain decomposition problem, note that if each polyomino in the tiling has min perimeter, then the problem is solved. In a tiling of an arbitrary domain, not all polyominos can have min perimeter, and solutions involve approximating the "all-min-perimeter" situation. Yackel-Meyer-Christou [8] found a simple formula for the min perimeter of a polyomino.

Theorem 1. **(Yackel-Meyer-Christou)** *The min perimeter of an n-polyomino is* $2\lceil 2\sqrt{n}\,\rceil$.

Idea. A polyomino has min perimeter if it is a square, or closely resembles a square. An n-polyomino has area n. If we shape this n-polyomino into a square having the same area, then the square has side $\sqrt{n}$ and perimeter $4\sqrt{n}$. So a lower bound for the min perimeter of an n-polyomino can be shown to be

$4\sqrt{n}$. But this lower bound is not always integer. It turns out that $2\lceil 2\sqrt{n}\rceil$ is a lower bound and is always attainable. □

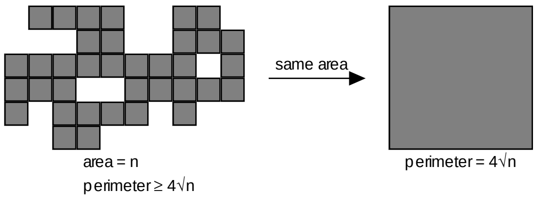

Figure 4: Relation between polyomino and square of same area.

3 Polyomino slices

The perimeter of a polyomino is related to its numbers of "subslices". A *slice* is a row or column containing squares (the squares need not be connected). A *subslice* is a maximal connected set of squares in a slice. A *slice-gap* is an absence of squares between subslices in a row or column. A slice is *convex* iff the set of squares in the slice is convex; the slice has no gaps. A polyomino is *slice-convex* iff every slice is convex.

Theorem 2. (**Polyomino subslices theorem**) *The perimeter of a polyomino is 2 times the number of subslices.*

Proof. Every subslice contributes 2 boundary edges. See Figure 5. □

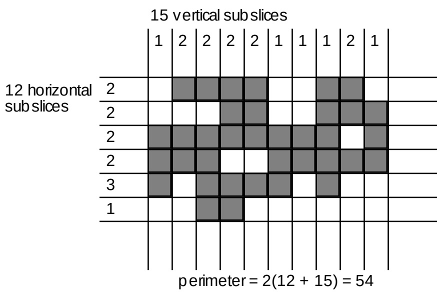

Figure 5: The perimeter of a polyomino is 2 times the number of subslices.

The following theorem follows immediately.

Theorem 3. (**Polyomino slices theorem**) *The perimeter of a slice-convex polyomino is* 2 *times the number of slices.*

4 Minimal polyiamond formula

We now begin consideration of the minimal polyiamonds that provide a lower bound for the optimal value of the following domain decomposition problem:

Problem 2. (**Domain decomposition problem with polyiamonds**) *Let* n *divide* A*. Tile a given set of* A *equilateral triangles with* n*-polyiamonds. What is the min total perimeter of the polyiamonds in such a tiling?*

The following parity theorem plays a crucial role in the minimal polyiamond formula.

Theorem 4. (**Polyiamond perimeter parity**) *The perimeter of an* n*-polyiamond has the same parity as* n*.*

Proof. This follows from the following claim, where the definition of *quasi-polyiamond* is similar to that of "polyiamond", except that the triangles need not be connected.

> Claim: The perimeter of an n-quasi-polyiamond has the same parity as n.

Proof of claim: Use induction. The base case is $n = 0$. There is 1 n-quasi-polyiamond, and its perimeter is 0.

Assume the statement is true for some $n \geq 0$. Note an $(n+1)$-quasi-polyiamond can be constructed by adding 1 triangle to an n-quasi-polyiamond. The *degree* of a triangle is the number of triangles adjacent to it. The degree of the new triangle determines the change in perimeter:

degree	edges created	edges destroyed	perimeter change
0	3	0	$+3$
1	2	1	$+1$
2	1	2	-1
3	0	3	-3

By the induction hypothesis, the old perimeter has the same parity as n. By the table above, the perimeter change is always odd. So the new perimeter has parity different from that of n. So the new perimeter has the same parity as $n+1$. ☐

We now come to the main result of this paper, the minimal polyiamond formula. The proof is short and uses the Polyiamond perimeter parity (Theorem 4) and the Minimal polyiamond bounds (Theorem 18), proved later. Polyiamonds that attain min perimeter are *minimal* (or *optimal*).

Theorem 5. **(Minimal polyiamond formula)** *The min perimeter* $\operatorname{minperim}(n)$ *of an n-polyiamond is whichever of* $\lceil\sqrt{6n}\rceil$ *or* $\lceil\sqrt{6n}\rceil+1$ *has the same parity as n.*

Proof. By the Minimal polyiamond bounds (Theorem 18),

$$\left\lceil\sqrt{6n}\right\rceil \leq \operatorname{minperim}(n) \leq \left\lceil\sqrt{6n}\right\rceil + 1.$$

The upper and lower bounds are integer and differ by exactly 1, so the bounds have different parity, and $\operatorname{minperim}(n)$ equals one of the bounds. By the polyiamond perimeter parity theorem, $\operatorname{minperim}(n)$ has the same parity as n. So the bound with the same parity as n is equal to $\operatorname{minperim}(n)$. ☐

See the appendix for a table of $\operatorname{minperim}(n)$. In the Minimal polyiamond bounds (Theorem 18), the upper and lower bounds are each attained about half the time. For example, if $1 \leq n \leq 10^6$, then $\operatorname{minperim}(n) = \lceil\sqrt{6n}\rceil$ for 500408 values of n. We give some motivation for why the bounds have the expression $\sqrt{6n}$.

Recall Yackel-Meyer-Christou's theorem that the min perimeter of an n-polyomino is $2\lceil 2\sqrt{n}\rceil$. This was motivated by shaping an n-polyomino into a square of the same area (see Figure 4). Consider shaping an n-polyiamond into a regular hexagon of the same area (see Figure 6). The polyiamond has area $A = n\sqrt{3}/4$, and it is easy to verify that the perimeter of the regular hexagon is $\sqrt{8\sqrt{3}A} = \sqrt{6n}$.

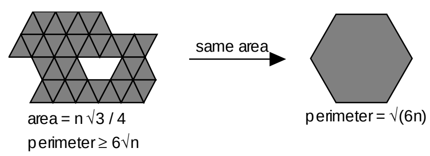

Figure 6: Relation between polyiamond and regular hexagon of same area.

5 Polyiamond slices

We generalize the slice approach used with polyominos. With polyominos, we have 2 kinds of slices: horizontal and vertical. But with polyiamonds, we have 3 kinds of slices: horizontal, antidiagonal, and diagonal ("HAD"). For brevity, we say that a polyiamond has *HAD slices (or dimensions)* (h, a, d) iff it has h horizontal slices, a antidiagonal slices, and d diagonal slices.

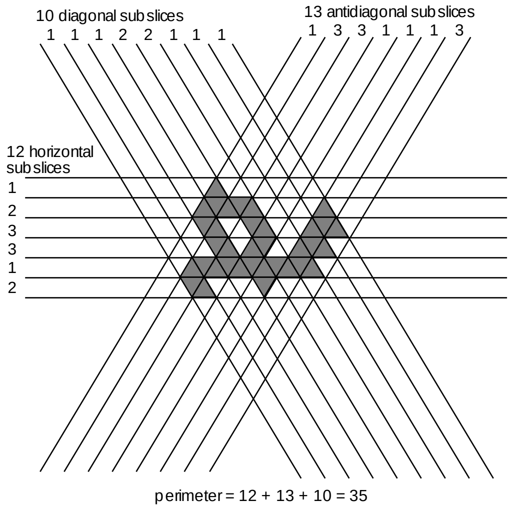

Figure 7: The perimeter of a polyiamond is the number of subslices.

Theorem 6. **(Polyiamond subslices theorem)** *The perimeter of a polyiamond is the number of subslices.*

Proof. See Figure 7. Note each of the 2 ends of every subslice is a boundary edge. Also, every boundary edge is the end of exactly 2 subslices. So the number of subslices equals the number of boundary edges, which equals the perimeter. □

The following theorem follows immediately.

Theorem 7. **(Polyiamond slices theorem)** *The perimeter of a slice-convex polyiamond is the number of slices.*

In order to construct maximal polyiamonds (polyiamonds of given perimeter and the most triangles), we start with polyiamonds of given HAD dimensions and the most triangles. Using the HAD capacity algorithm (Theorem 8) and the HAD capacity formula (Theorem 9), we show that size (number of triangles) is maximized for a given perimeter $p = h + a + d$ by "balancing" the dimensions (choosing them as close together as possible).

To simplify the presentation, we assume for the remainder of the paper that the HAD dimensions satisfy $h \le a \le d$. For example, the polyiamond of Figure 7 satisfies these inequalities because its HAD dimensions are $(h, a, d) = (6, 7, 8)$. For an arbitrary polyiamond, it is easy to see that the inequalities $h \le a \le d$ can be attained by rotation and reflection. We also assume that $p \ne 1$ or 2 because there is no polyiamond with these perimeters.

Theorem 8. (**HAD capacity algorithm**) *To construct a polyiamond with given HAD dimensions* (h, a, d) *and the most triangles, do the following:*

- *Draw a parallelogram with HAD dimensions* $(a + d, a, d)$.
- *Pick the* h *horizontal slices with the most triangles.*

Such a polyiamond is unique, ignoring rotation and reflection.

Proof. See Figure 8. Note that a polyiamond with a antidiagonal slices and d diagonal slices fits inside a unique parallelogram with a antidiagonal slices and d diagonal slices (this parallelogram is the "AD parallelogram hull", analogous to the convex hull). It is easily seen that $h \le a + d$. Note that for the number of triangles to be maximized, the polyiamond must have no gaps. Constructing the polyiamond as described ensures no gaps and ensures uniqueness. □

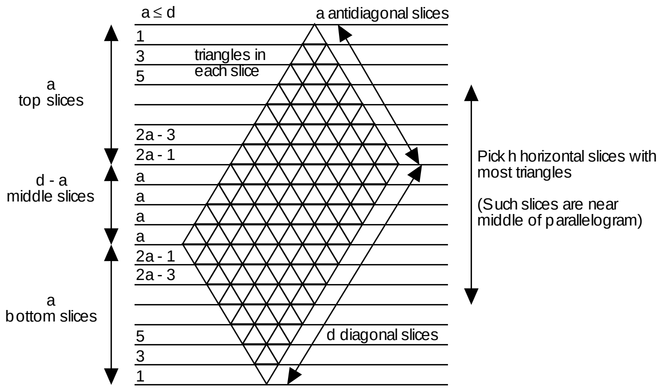

Figure 8: Constructing a polyiamond with HAD dimensions (h, a, d) and the most triangles.

Theorem 9. (**HAD capacity formula**) *Let $h \le a \le d$. Let*

$$A = ad + ah + dh - \frac{1}{2}(a^2 + d^2 + h^2).$$

The max number of triangles in a polyiamond with HAD dimensions (h, a, d) is

$$\text{capacity}\,(h, a, d) = \begin{cases} 2ah & h - (d - a) < 0 \\ A & h - (d - a) \ge 0 \text{ and is even} \\ A - 1/2 & h - (d - a) \ge 0 \text{ and is odd} \end{cases}$$

Proof. There are 3 cases.

- Case: $h - (d - a) < 0$. So $h < d - a$. See Figure 8. The $d - a$ horizontal slices in the middle of the parallelogram have the most triangles; each slice has $2a$ triangles. Pick h of these slices to construct a polyiamond with $2ah$ triangles.

- Case: $h - (d - a) \ge 0$ and is even. Note $h \ge d - a$. See Figure 9. The number of triangles in the polyiamond is

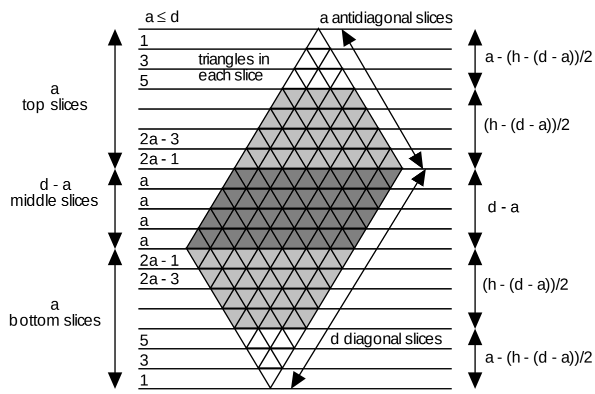

Figure 9: Counting triangles in a polyiamond with HAD dimensions (h, a, d) and the most triangles, where $h - (d - a) \geq 0$ and is even.

$$
\begin{aligned}
& 2a(d-a) + 2\left(a^2 - \left(a - \frac{h-(d-a)}{2}\right)^2\right) \\
= \quad & ad + ah + dh - \frac{1}{2}(a^2 + d^2 + h^2).
\end{aligned}
$$

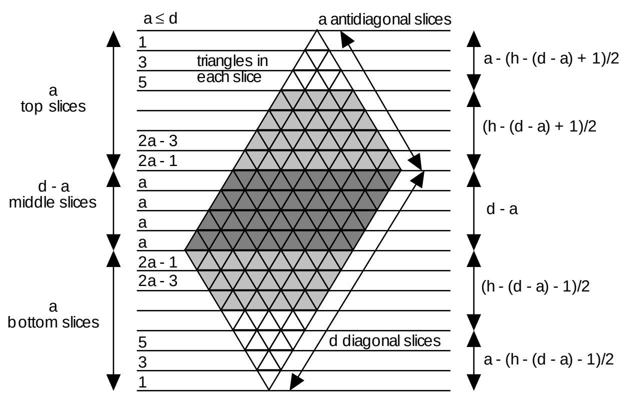

Figure 10: Counting triangles in a polyiamond with HAD dimensions (h, a, d) and the most triangles, where $h - (d - a) \geq 0$ and is even.

- Case: $h - (d - a) \geq 0$ and is odd. Note $h \geq d - a$. See Figure 10. The number of triangles in the polyiamond is

$$
\begin{aligned}
& 2a(d-a) \\
& + \left(a^2 - \left(a - \frac{h-(d-a)+1}{2}\right)^2\right) + \left(a^2 - \left(a - \frac{h-(d-a)-1}{2}\right)^2\right) \\
= \quad & ad + ah + dh - \frac{1}{2}(a^2 + d^2 + h^2) - \frac{1}{2}.
\end{aligned}
$$

$\square$

6 Maximal polyiamond formula

Theorem 10. *(***Maximal polyiamond dimensions and capacity theorem***)*

- *Let* capacity (p) *denote the max number of triangles in a polyiamond with perimeter p. Let* $p \neq 1$ *or* 2 *because there is no polyiamond with these perimeters.*
- *Let* (h, a, d) *be the HAD dimensions of such a polyiamond. (Without loss of generality, let* $h \leq a \leq d$*. We can rotate and reflect the polyiamond if necessary to get these inequalities.)*

Then capacity (p) *and* (h, a, d) *are as follows.*

p	h	a	d	capacity (p)
$3q$	q	q	q	$\lfloor \frac{3}{2}q^2 \rfloor$
$3q+1$	q	q	$q+1$	$\lfloor \frac{3}{2}q^2 + q - \frac{1}{2} \rfloor$
$3q+2$	q	$q+1$	$q+1$	$\lfloor \frac{3}{2}q^2 + 2q \rfloor$

Also, the capacity formulas in the preceding table can be consolidated as follows:

$$\text{capacity}\,(p) = \left\lfloor \frac{p^2}{6} - \frac{4}{6}\delta_3 \right\rfloor, \quad \delta_3 = \begin{cases} 0 & \textit{else if } p \equiv 0 \pmod 3 \\ 1 & \textit{else} \end{cases}$$

Proof. We will give an integer programming problem that maximizes the number of triangles in a polyiamond with perimeter exactly p. We note the following:

- To maximize the number of triangles, the polyiamond should have no gaps. By the Polyiamond slices theorem (Theorem 7), $h + a + d = p$.
- Without loss of generality, let $h \leq a \leq d$ (rotate and reflect the polyiamond to get these inequalities). We use these inequalities in the proof.

- We must have $h \geq d - a$ in a maximal polyiamond, by the following reasoning. If $h < d - a$, then there is some diagonal slice that does not intersect the h horizontal slices of the polyiamond (see Figure 8). We can remove this diagonal slice and add a horizontal slice to the polyiamond. The polyiamond now has HAD dimensions $(h+1, a, d-1)$, has more triangles, and has the same perimeter, contradicting maximality.

- Because $h \geq d - a$, by the HAD capacity formula (Theorem 9), the number of triangles in the polyiamond is as follows, where $c(h, a, d)$ is a correction term.

$$\begin{aligned}
\text{number of triangles} &= ad + ah + hd - \frac{1}{2}(a^2 + d^2 + h^2) + c(h, a, d). \\
c(h, a, d) &= \begin{cases} 0 & h - (d - a) \geq 0 \text{ and is even} \\ -1/2 & h - (d - a) \geq 0 \text{ and is odd} \end{cases}
\end{aligned}$$

The problem of maximizing the number of triangles in a polyiamond with perimeter p can therefore be expressed as follows:

$$\left[\begin{array}{llcll}
\max & ad + ah + hd - \frac{1}{2}(a^2 + d^2 + h^2) & + & c(h, a, d) & \\
\text{s.t.} & h + a + d & = & p & \\
 & h & \leq & a & \leq \ d \\
 & h, a, d & \in & \mathbf{Z} &
\end{array}\right]$$

Simplify the objective function by using the constraint $h + a + d = p$.

$$\left[\begin{array}{llcll}
\max & \frac{1}{2}p^2 - (a^2 + d^2 + h^2) & + & c(h, a, d) & \\
\text{s.t.} & h + a + d & = & p & \\
 & h & \leq & a & \leq \ d \\
 & h, a, d & \in & \mathbf{Z} &
\end{array}\right]$$

Express the problem in terms of $x^T = (h, a, d)$ and $e^T = (1, 1, 1)$, and consider the relaxed problem obtained by dropping the constraint $h \leq a \leq d$.

$$\left[\begin{array}{llcl}
\max & \frac{1}{2}p^2 - x^T x & + & c(x) \\
\text{s.t.} & e^T x & = & p \\
 & x & \in & \mathbf{Z^3}
\end{array}\right]$$

Bring out the constant summand $p^2/2$ and the constant factor -1, and change the max to a min.

$$\frac{p^2}{2} - \left[\begin{array}{llll} \min & x^T x & + & c(x) \\ \text{s.t.} & e^T x & = & p \\ & x & \in & \mathbf{Z}^3 \end{array} \right]$$

Drop the integer constraints (it turns out that we will be able to get integer solutions without them). We have the following relaxed problem:

$$\text{RP} = \frac{p^2}{2} - \left[\begin{array}{llll} \min & x^T x & + & c(x) \\ \text{s.t.} & e^T x & = & p \end{array} \right]$$

We will derive an alternative expression of the correction term $c(x) = c(h, a, d)$. At the beginning of this proof, we derived $h \geq d - a$. So $h - (d - a) \geq 0$. Note $h - (d - a)$ is even iff $h + a + d = e^T x = p$ is even. We can express the correction term $c(h, a, d)$ as follows:

$$c(h, a, d) = \left\{ \begin{array}{rl} 0 & h - (d - a) \geq 0 \text{ and is even} \\ -1/2 & h - (d - a) \geq 0 \text{ and is odd} \end{array} \right. = \left\{ \begin{array}{rl} 0 & p \text{ even} \\ -1/2 & p \text{ odd} \end{array} \right.$$

The relaxed problem RP branches into 2 relaxed problems, one for the case p odd and one for the case p even.

$$\begin{array}{lll} \text{RP\_ODD} & = & \frac{p^2}{2} - \left[\begin{array}{lllll} \min & x^T x & - & 1/2 & \\ \text{s.t.} & e^T x & = & p & \text{odd} \end{array} \right] \\ \text{RP\_EVEN} & = & \frac{p^2}{2} - \left[\begin{array}{lllll} \min & x^T x & & & \\ \text{s.t.} & e^T x & = & p & \text{even} \end{array} \right] \end{array}$$

In each of these relaxed problems, the objective function is strictly convex. So any solution of these relaxed problems (and the related restricted problems considered below) is unique.

There are 3 cases: p can have the form $3q$, $3q + 1$, or $3q + 2$. In each case, it turns out that RP_ODD and RP_EVEN have solutions of the same form when expressed in terms of p.

For example, let $p = 3q + 2$. It turns out that if p is odd, then $x_{\text{odd}} = (q, q + 1, q + 1)$ solves RP_ODD. If p is even, then $x_{\text{even}} = (q, q + 1, q + 1)$ solves RP_EVEN. Note $x_{\text{odd}} = x_{\text{even}}$, in the sense that they have the same form.

- Case: $p = 3q$. Consider the relaxed problem. Both RP_ODD and RP_EVEN have the solution $x = (q, q, q)$, which also solves the initial integer problem. The optimal value is

$$\text{capacity}\,(p) = \text{optimal value} = \frac{p^2}{2} - x^T x + c(x) = \frac{3}{2}q^2 + c(x).$$

  To express the optimal value in a simpler form, note that if p is even, then $c(x) = 0$ and $p^2/2 - x^T x$ is an integer, and if p is odd, then $c(x) = -1/2$ and $p^2/2 - x^T x$ is an integer plus $1/2$. So the optimal value is

$$\left\lfloor \frac{3}{2}q^2 \right\rfloor.$$

- Case: $p = 3q + 1$. Because p has the form $p = 3q + 1$ and $h \le a \le d$, we must have $h \le d - 1$. There are 2 subcases.

  - Subcase: $h = d - 1$. Because p has the form $p = 3q + 1$, we must also have $a = d - 1$. Add these constraints to RP_ODD and RP_EVEN. Both problems have the solution $x = (q, q, q+1)$, which also solves the integer problem. The optimal value is

$$\left\lfloor \frac{3}{2}q^2 + q - \frac{1}{2} \right\rfloor.$$

  - Subcase: $h \le d-2$. Add this constraint to RP_ODD and RP_EVEN. Both problems have the solution $x = (q - 1, q + 1, q + 1)$, which also solves the integer problem. The optimal value is

$$\left\lfloor \frac{3}{2}q^2 + q - \frac{5}{2} \right\rfloor.$$

  The subcase $h = d - 1$ yields the solution because it gives the larger value.

- Case: $p = 3q + 2$. Again, $h \leq a \leq d$ implies $h \leq d - 1$. Add this constraint to RP_ODD and RP_EVEN. Both problems have the solution $x = (q, q+1, q+1)$, which also solves the integer problem. The optimal value is

$$\left\lfloor \frac{3}{2}q^2 + 2q \right\rfloor .$$

Summarizing all the cases, we get the table stated in the theorem. The statement about the consolidated capacity formula follows from the following calculations, in which

$$\delta_3 = \begin{cases} 0 & p \equiv 0 \pmod 3 \\ 1 & \text{else} \end{cases}$$

p	p^2	$\frac{p^2}{6}$	δ_3	$\frac{p^2}{6} - \frac{4}{6}\delta_3$
$3q$	$9q^2$	$\frac{3}{2}q^2$	0	$\frac{3}{2}q^2$
$3q+1$	$9q^2 + 6q + 1$	$\frac{3}{2}q^2 + q + \frac{1}{6}$	1	$\frac{3}{2}q^2 + q - \frac{1}{2}$
$3q+2$	$9q^2 + 12q + 4$	$\frac{3}{2}q^2 + 2q + \frac{2}{3}$	1	$\frac{3}{2}q^2 + 2q$

□

A polyiamond with HAD dimensions (h, a, d) is *balanced* iff it is slice-convex and h, a, d differ from one another by at most 1.

Theorem 11. **(Maximal-balanced equivalence theorem)** *A polyiamond is maximal iff it is balanced.*

Proof. Use the the Maximal polyiamond dimensions and capacity theorem (Theorem 10). □

Theorem 12. **(Maximal polyiamond formula)** *The max number of triangles in a polyiamond with perimeter p is*

$$\text{capacity}(p) = \text{round}\left(\frac{p^2}{6}\right) - \begin{cases} 0 & \textit{else if } p \equiv 0 \pmod 6 \\ 1 & \textit{else} \end{cases}$$

Recall that there is no polyiamond with perimeter 1 or 2.

Proof. Let

$$\delta_i = \begin{cases} 0 & p \equiv 0 \pmod{i} \\ 1 & \text{else} \end{cases}$$

Use the the Maximal polyiamond dimensions and capacity theorem (Theorem 10):

$$\text{capacity}\,(p) = \left\lfloor \frac{p^2}{6} - \frac{4}{6}\delta_3 \right\rfloor.$$

Let $p = 6q + r$, where $r = 0, \ldots, 5$. Note the following equivalences.

$$\begin{aligned}
& \left\lfloor \frac{p^2}{6} - \frac{4}{6}\delta_3 \right\rfloor &= \ & \text{round}\left(\frac{p^2}{6}\right) - \delta_6 \\
\iff & \left\lfloor 6q^2 + 2qr + \frac{r^2}{6} - \frac{4}{6}\delta_3 \right\rfloor &= \ & \text{round}\left(6q^2 + 2qr + \frac{r^2}{6}\right) - \delta_6 \\
\iff & 6q^2 + 2qr + \left\lfloor \frac{r^2}{6} - \frac{4}{6}\delta_3 \right\rfloor &= \ & 6q^2 + 2qr + \text{round}\left(\frac{r^2}{6}\right) - \delta_6 \\
\iff & \left\lfloor \frac{r^2}{6} - \frac{4}{6}\delta_3 \right\rfloor &= \ & \text{round}\left(\frac{r^2}{6}\right) - \delta_6.
\end{aligned}$$

The last equality is easily verified by considering the cases $r = 0, \ldots, 5$.

r	$\frac{r^2}{6}$	δ_3	$\frac{r^2}{6} - \frac{4}{6}\delta_3$	$\left\lfloor \frac{r^2}{6} - \frac{4}{6}\delta_3 \right\rfloor$	$\text{round}\left(\frac{r^2}{6}\right)$	δ_6	$\text{round}\left(\frac{r^2}{6}\right) - \delta_6$
0	0	0	0	0	0	0	0
1	1/6	1	−3/6	−1	0	1	−1
2	4/6	1	0	0	1	1	0
3	9/6	0	9/6	1	2	1	1
4	16/6	1	2	2	3	1	2
5	25/6	1	21/6	3	4	1	3

□

See the appendix for a table of capacity (p). We can interpret the maximal polyiamond formula as approximating a regular hexagon, by the following reasoning. Note $p^2/6$ is integer iff p is a multiple of 6, say $p = 6k$. A regular hexagon of side $p = 6k$ has side k and has $6k^2 = p^2/6$ triangles. If p is not a multiple of 6, then we cannot construct a regular hexagon and we have to subtract 1 triangle from the polyiamond.

7 Maximal polyiamond algorithm

We give 2 versions of the maximal polyiamond algorithm, a slice version and a spiral version. The slice version contains the proof. The spiral version is an alternate approach.

Theorem 13. **(Maximal polyiamond slice algorithm)** *See Figure 11. To construct a polyiamond with perimeter p and the most triangles, do the following:*

- *If $p = 1$ or 2, stop. There is no polyiamond with this perimeter.*
- *Find the HAD dimensions (h, a, d) in the following table.*

p	h	a	d
$3q$	q	q	q
$3q+1$	q	q	$q+1$
$3q+2$	q	$q+1$	$q+1$

- *Draw a parallelogram with HAD dimensions $(a + d, a, d)$.*
- *Pick the h horizontal slices with the most triangles.*

Such a polyiamond is unique, ignoring rotation and reflection.

Proof. Use the Maximal-balanced equivalence theorem (Theorem 11) and the HAD capacity algorithm (Theorem 8). □

An alternative algorithm that produces polyiamonds of the most triangles is as follows; from now on, "maximal polyiamond algorithm" will refer to this spiral version:

Theorem 14. **(Maximal polyiamond spiral algorithm)** *See Figure 12. To construct a polyiamond with perimeter p and the most triangles, follow the spiral and stop at the triangle labeled with perimeter p.*

In Figure 12, the number in a triangle represents perimeter of the polyiamond constructed up to that point. The number of triangles in the polyiamond constructed is just the number of triangles from the start of the spiral up to that point. A triangle has a number iff it represents the end of a

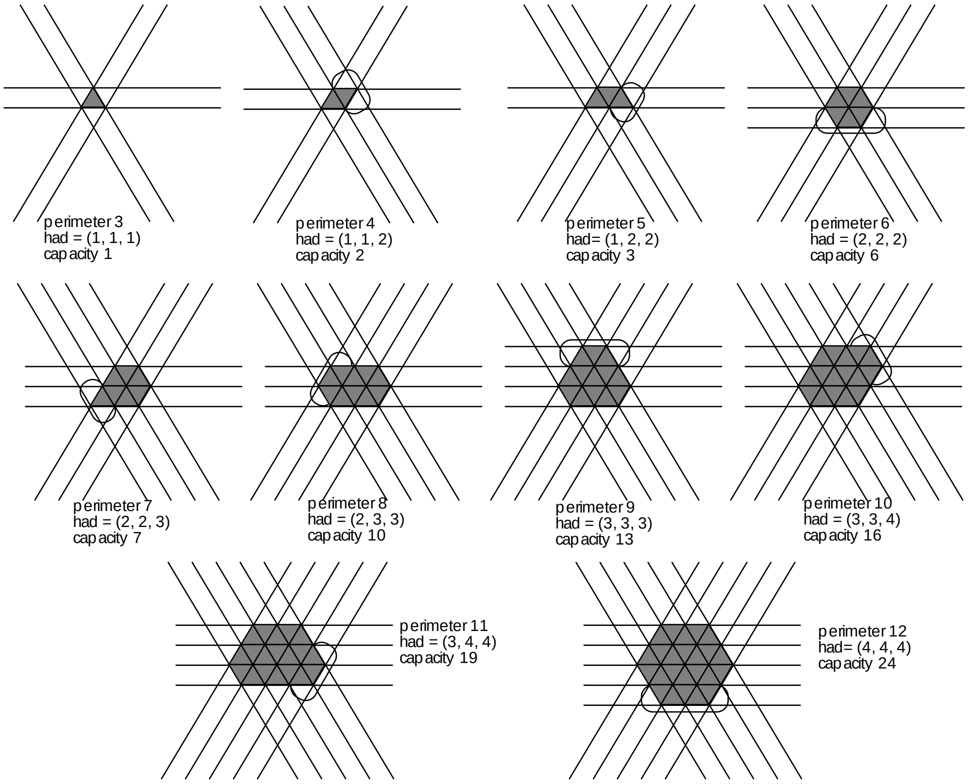

Figure 11: Maximal polyiamond slice algorithm. The circled triangles at the polyiamond with perimeter p are the triangles that have been added to the polyiamond with perimeter $p-1$.

slice. Not all triangles have numbers, because we are interested in only the polyiamonds of a given perimeter and with the most triangles.

The triangles labeled with numbers are shaded to make them stand out from the other triangles. There is a pattern to the shaded triangles once we travel far enough along the spiral.

8 Minimal polyiamond algorithm

The minimal polyiamond algorithm allows us to prove many results in this paper. Its proof uses the maximal polyiamond algorithm.

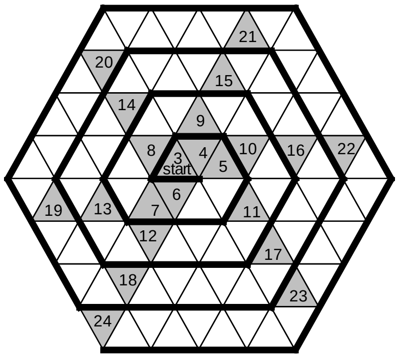

Figure 12: Maximal polyiamond spiral algorithm. To construct a polyiamond with perimeter p of the most triangles, follow the spiral and stop at the triangle labeled with perimeter p.

Theorem 15. **(Minimal polyiamond algorithm)** *See Figure 13. To construct a polyiamond with min perimeter and n triangles, follow the spiral for n triangles.*

Proof. We must prove that the polyiamond constructed by the above algorithm is minimal. This follows from the following claim.

> Claim: Let minperim(n) be the min perimeter of an n-polyiamond.
>
> Let reqperim (n) be the min perimeter of a polyiamond with n or more triangles. (Think of reqperim (n) as the "perimeter required for n or more triangles".)
>
> Then minperim(n) is whichever of reqperim (n) or reqperim (n)+ 1 has the same parity as n.

Proof of claim: Throughout this proof, for brevity, let $r =$ reqperim (n). Note minperim(n) $\geq r$. By the Polyiamond perimeter parity (Theorem 4), minperim(n) has the same parity as n. There are 2 cases. If r has the same parity as n, then minperim(n) $\geq r$. Else, $r + 1$ has the same parity as n, and minperim(n) $\geq r + 1$. So minperim(n) is bounded below by whichever of r or $r + 1$ has the same parity as n.

We prove that the lower bound is always attainable by constructing an n-polyiamond whose perimeter is whichever of r or $r + 1$ has the same parity as n. Below, we explain how the algorithm described in the theorem gives such a construction. Note that the minimal polyiamond algorithm spiral in Figure 13 is the same as the maximal polyiamond algorithm spiral in Figure 12. The only differences between these figures are the shading and labels of some triangles.

Consider what happens when we follow the minimal polyiamond algorithm spiral in Figure 13 for n triangles. If we interpret this spiral in terms of the Maximal polyiamond spiral algorithm (Theorem 14), then along the way to n triangles, we will construct the unique polyiamond with perimeter $r-1$ and capacity $(r-1)$ triangles. Let us stop at this point. Consider the perimeter and number of triangles of the polyiamond constructed by following the spiral for i more triangles, where $i + \text{capacity}\,(r-1) \leq \text{capacity}\,(r)$:

- If we follow the spiral for 1 more triangle, the number of triangles will be $1 + \text{capacity}\,(r-1)$, and the perimeter will be r.
- If we follow the spiral for 2 more triangles, the number of triangles will be $2 + \text{capacity}\,(r-1)$, and the perimeter will be $r+1$ (a gap has been introduced).
- If we follow the spiral for 3 more triangles, the number of triangles will be $3 + \text{capacity}\,(r-1)$, and the perimeter will be r (a gap has been filled).
- In general, if we follow the spiral for i more triangles, the number of triangles will be $i + \text{capacity}\,(r-1)$, and the perimeter will be r if i is odd, and will be $r+1$ else. □

See the appendix for a table of minperim(n). The idea of the minimal polyiamond algorithm is to add triangles to a regular hexagon of side k to construct a regular hexagon of side $k+1$; start with 1 triangle, and at each step, add a triangle in the clockwise direction so that it is adjacent to the triangle added in the previous step. Note that reaching or turning a corner (at the vertices of the hexagon) adds a slice to the polyiamond constructed. Note that all triangles added along a side of the hexagon are in the same slice; this keeps the perimeter small enough to be minimal.

Note that the balanced HAD polyiamonds constitute the unique minimax configurations; i.e., they are the only polyiamonds that are both minimal and maximal.

Figure 13 keeps track of the perimeters of the polyiamonds constructed. White triangles (except for the first) add 1 to the perimeter, and black trian-

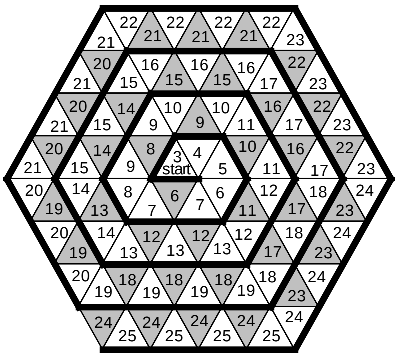

Figure 13: Minimal polyiamond algorithm. To construct a polyiamond with min perimeter and n triangles, follow the spiral for n triangles. The number in a triangle is the perimeter of the polyiamond constructed. White triangles (except for the first) indicate an increase of 1 in perimeter, and black triangles indicate a decrease of 1 in perimeter.

gles subtract 1 from the perimeter. The black and white triangles are alternating except at the turns of the spiral, where 2 consecutive white triangles appear. We could call the minimal polyiamond algorithm the "diamondback rattlesnake algorithm", because of the resemblance of the black and white triangles to the markings of a curled-up diamondback rattlesnake.

9 Figures related to the minimal polyiamond algorithm

The following figures follow immediately from the Minimal polyiamond algorithm (Theorem 15). We will use them to prove the minimal polyiamond bounds and other theorems. We put the figures here for easy reference.

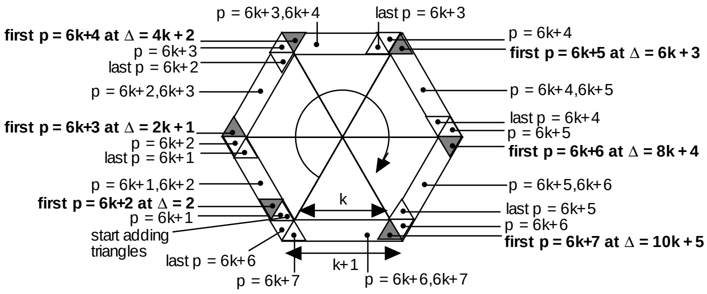

Figure 14: Appearances of perimeters in minimal polyiamond algorithm.

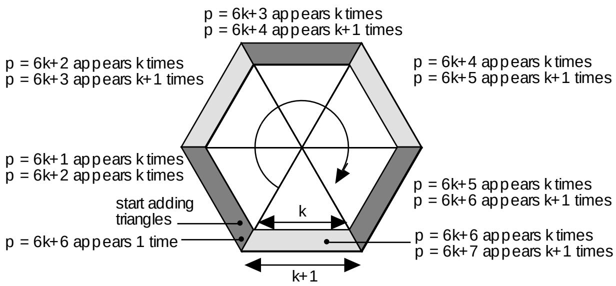

Figure 15: Frequency of perimeters in minimal polyiamond algorithm.

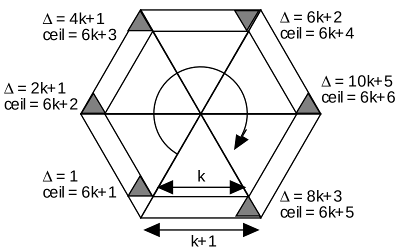

Figure 16: First appearance of values of $\lceil\sqrt{6n}\rceil$ for triangles added in minimal polyiamond algorithm.

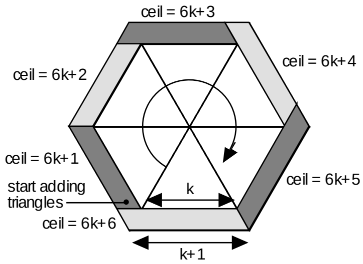

Figure 17: Distribution of values of $\lceil\sqrt{6n}\rceil$ for triangles added in minimal polyiamond algorithm.

Theorem 16. (**Minimal polyiamond intermediate perimeter theorem**) *Let $k \geq 1$. In the Minimal polyiamond algorithm (Theorem 15), when we add Δ triangles to a regular hexagon of side k, the perimeters of the polyiamonds constructed are as follows:*

$\Delta_{\min}$	$\leq$	Δ	$\leq$	$\Delta_{\max}$	minperim() $6k^2 + \Delta_{\min}$	minperim() $6k^2 + \Delta_{\max}$	p
1	$\leq$	Δ	$\leq$	$2k$	$6k+1$	$6k+2$	$6k+1, 6k+2$
$2k+1$	$\leq$	Δ	$\leq$	$4k+1$	$6k+3$	$6k+3$	$6k+2, 6k+3$
$4k+2$	$\leq$	Δ	$\leq$	$6k+2$	$6k+4$	$6k+4$	$6k+3, 6k+4$
$6k+3$	$\leq$	Δ	$\leq$	$8k+3$	$6k+5$	$6k+5$	$6k+4, 6k+5$
$8k+4$	$\leq$	Δ	$\leq$	$10k+4$	$6k+6$	$6k+6$	$6k+5, 6k+6$
$10k+5$	$\leq$	Δ	$\leq$	$12k+6$	$6k+7$	$6k+6$	$6k+6, 6k+7$

Proof. See Figure 14 and Figure 15. □

Theorem 17. (**Minimal polyiamond ceiling theorem**) *Let $k \geq 1$. In the Minimal polyiamond algorithm (Theorem 15), when we add Δ triangles to a regular hexagon of side k, and if $n = 6k^2 + \Delta$, then the values of $\lceil \sqrt{6n} \rceil$ are as follows:*

Δ					$\lceil \sqrt{6n} \rceil$
1	$\leq$	Δ	$\leq$	$2k$	$6k+1$
$2k+1$	$\leq$	Δ	$\leq$	$4k$	$6k+2$
$4k+1$	$\leq$	Δ	$\leq$	$6k+1$	$6k+3$
$6k+2$	$\leq$	Δ	$\leq$	$8k+2$	$6k+4$
$8k+3$	$\leq$	Δ	$\leq$	$10k+4$	$6k+5$
$10k+5$	$\leq$	Δ	$\leq$	$12k+6$	$6k+6$

Proof. See Figure 16 and Figure 17. The intervals of Δ in the table follow from the following abbreviated calculations with $i = 0, \ldots, 5$.

$$\begin{array}{lrcccl} & & & \lceil \sqrt{6n} \rceil & = & 6k+i \\ \iff & 6k+(i-1) & < & \sqrt{6n} & \leq & 6k+i \\ \iff & 2k(i-1)+\dfrac{(i-1)^2}{6} & < & \Delta & \leq & 2ki+\dfrac{i^2}{6}. \end{array}$$

□

10 Minimal polyiamond bounds

When we proved the Minimal polyiamond formula (Theorem 5), we assumed the minimal polyiamond bounds, which we now prove.

Theorem 18. (**Minimal polyiamond bounds**) *The min perimeter* $\operatorname{minperim}(n)$ *of an n-polyiamond satisfies the following bounds:*

$$\left\lceil \sqrt{6n} \right\rceil \leq \operatorname{minperim}(n) \leq \left\lceil \sqrt{6n} \right\rceil + 1.$$

Proof. Use the Minimal polyiamond algorithm (Theorem 15). There are 2 cases.

- Case: $n \leq 6$. It is easy to verify the bounds for these values of n.
- Case: $n > 6$. Compare the tables in the Minimal polyiamond intermediate perimeter theorem (Theorem 16) and the Minimal polyiamond

ceiling theorem (Theorem 17). Recall that Δ is the number of triangles added to a regular hexagon of side k, and $n = 6k^2 + \Delta$. Note that the intervals of Δ in the tables are very similar; the endpoints differ by at most 1. It is easy to check that the bounds hold for all values of Δ. □

11 Miscellaneous consequences of the minimal polyiamond algorithm

Theorem 19. **(Minimal polyiamond first-appearance-of-min-perimeter theorem)** *In the Minimal polyiamond algorithm (Theorem 15), a triangle contains the first appearance of a perimeter iff the triangle is "immediately after a turn of the spiral".*

Proof. See Figure 14. □

Theorem 20. **(Minimal polyiamond 2-apart appearances theorem)** *In the Minimal polyiamond algorithm (Theorem 15), if a perimeter p appears more than once, and if it appears at triangle Δ, then it appears at triangle $\Delta - 2$ or $\Delta + 2$.*

Proof. See Figure 13 or Figure 14. □

Theorem 21. **(Minimal polyiamond frequency theorem)** *In the Minimal polyiamond algorithm (Theorem 15), every perimeter p appears $\lfloor p/3 \rfloor$ times.*

Proof. This follows from the following claim:

> Claim: In the minimal polyiamond algorithm, at the completion of a hexagon of side k, the perimeter $6k+1$ has appeared k times, and each smaller perimeter p has appeared $\lfloor p/3 \rfloor$ times.

Proof of claim: Use induction. The claim is true for perimeters $p = 0$, 1, 2, and it is also true for a hexagon of side $k = 1$, with its perimeters $3, \ldots, 7$ (see Figure 13). Assume the claim is true for some $k \geq 1$. According to the minimal polyiamond algorithm, add triangles to a hexagon of side k to form a hexagon of side $k + 1$ (see Figure 14). We get the following table showing the number of appearances of certain perimeters during the addition of the triangles (see Figure 15).

p	appearances	
$6k+1$	k	
$6k+2$	$2k$	$= \lfloor p/3 \rfloor$
$6k+3$	$2k+1$	$= \lfloor p/3 \rfloor$
$6k+4$	$2k+1$	$= \lfloor p/3 \rfloor$
$6k+5$	$2k+1$	$= \lfloor p/3 \rfloor$
$6k+6$	$2k+2$	$= \lfloor p/3 \rfloor$
$6k+7$	$k+1$	

Only the perimeter $6k+1$ in the above table needs some explanation. The table shows that the perimeter $6k+1$ appears k times during the addition of triangles to a hexagon of side k to form a hexagon of side $k+1$. By the induction hypothesis, the perimeter $6k+1$ appears k times in the hexagon of side k. So the perimeter $6k+1$ appears a total of $2k = \lfloor (6k+1)/3 \rfloor$ times.

Also by the induction hypothesis, each perimeter $p \leq 6k$ appears $\lfloor p/3 \rfloor$ times. By the above table and reasoning, each perimeter $p \leq 6k+6 = 6(k+1)$ appears $\lfloor p/3 \rfloor$ times.

Also, the table shows that the perimeter $6k+7 = 6(k+1)+1$ appears $k+1$ times in the hexagon of side $k+1$. So the claim is proved. □

12 Capacity generating function

Recall that capacity (p) was defined to be the max number of triangles in a polyiamond with perimeter p. However, there is no polyiamond with perimeter 1 or 2, and so capacity (p) is undefined for these perimeters.

When we consider sequences a_n and the corresponding generating functions, it is nice to have a_n defined for all values of n. Let us make the following generalized definition: capacity (p) is the max number of triangles in a polyiamond with perimeter *at most* p. This definition agrees with the old one, except that now capacity $(1) =$ capacity $(2) = 0$. Also, the the Maximal polyiamond formula (Theorem 12) is still valid after a slight modification (see the appendix).

The following conjecture was motivated by [6].

Conjecture 1. *(***Capacity generating function conjecture***)*

$$\sum_{p=0}^{\infty} \text{capacity}\,(p)\, x^p = \frac{x^3(1+x^2)(1+x-x^2)}{(1-x)(1-x^2)(1-x^3)}.$$

13 For further research

Having developed the theory of minimal polyiamonds, we will investigate how this theory can be employed in the solution of the domain decomposition problem with polyiamonds. Approaches will be developed that are related to those that have been shown to be effective in the domain decomposition problem with polyominos.

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14 Appendix

The following table gives values of 2 sequences in the text. See Figure 12 and Figure 13.

- minperim(n) is the min perimeter of an n-polyiamond. minperim(n) is whichever of $\lceil\sqrt{6n}\rceil$ or $\lceil\sqrt{6n}\rceil + 1$ has the same parity as n.
- capacity (p) is the max number of triangles in a polyiamond with perimeter at most p (this is the generalized definition of capacity (p), discussed in the section about the capacity generating function). Included in the table are the HAD dimensions (h, a, d) of such a polyiamond.

$$\text{capacity}\,(p) = \text{round}\left(\frac{p^2}{6}\right) - \begin{cases} 0 & \text{if } p = 1 \\ 0 & \text{else if } p \equiv 0 \pmod 6 \\ 1 & \text{else} \end{cases}$$

The left side of the table, indexed by n, is independent of the right side of the table, indexed by p; the values between the 2 sides are not related. The (h, a, d) columns are to be used with only the capacity (p) column.

n	minperim(n)	p	capacity (p)	h	a	d
0	0	0	0	0	0	0
1	3	1	0	0	0	0
2	4	2	0	0	0	0
3	5	3	1	1	1	1
4	6	4	2	1	1	2
5	7	5	3	1	2	2
6	6	6	6	2	2	2
7	7	7	7	2	2	3
8	8	8	10	2	3	3
9	9	9	13	3	3	3
10	8	10	16	3	3	4
11	9	11	19	3	4	4
12	10	12	24	4	4	4
13	9	13	27	4	4	5
14	10	14	32	4	5	5
15	11	15	37	5	5	5
16	10	16	42	5	5	6
17	11	17	47	5	6	6
18	12	18	54	6	6	6
19	11	19	59	6	6	7
20	12	20	66	6	7	7
21	13	21	73	7	7	7
22	12	22	80	7	7	8
23	13	23	87	7	8	8
24	12	24	96	8	8	8