Solution of General Linear Complementarity Problems via Nondifferentiable Concave Minimization *

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Dedicated to Professor Hoang Tuy on the Occasion of His Seventieth Birthday

Abstract

Finite termination, at point satisfying the minimum principle necessary optimality condition, is established for a stepless (no line search) successive linearization algorithm (SLA) for minimizing a nondifferentiable concave function on a polyhedral set. The SLA is then applied to the general linear complementarity problem (LCP), formulated as minimizing a piecewise-linear concave error function on the usual polyhedral feasible region defining the LCP. When the feasible region is nonempty, the concave error function always has a global minimum at a vertex, and the minimum is zero if and only if the LCP is solvable. The SLA terminates at a solution or stationary point of the problem in a finite number of steps. A special case of the proposed algorithm [8] solved without failure 80 consecutive cases of the LCP formulation of the knapsack feasibilty problem, ranging in size between 10 and 3000.

1 Introduction

We consider the classical linear complementarity problem (LCP) [4, 12, 5]

$$0 \leq x \perp Mx + q \geq 0, \tag{1}$$

where $\perp$ denotes orthogonality, and no assumptions are made on the $n \times n$ real matrix M or the $n \times 1$ real vector q defining the problem. This NP-complete problem [3], which may not have a solution, is easily shown to be equivalent to the following minimization of a piecewise-linear concave function on the polyhedral set defining the LCP [7, Lemma 1]

$$0 = \min_x \{e'(x - (x - Mx - q)_+) \Big| Mx + q \geq 0,\ x \geq 0\}, \tag{2}$$

where e is a column vector of ones and z_+ denotes the component-wise maximum of z_i and 0 for a vector z. This problem in turn can be rewritten as:

$$0 = \min_x \{e' \text{minimum}\{x, Mx + q\} \Big| Mx + q \geq 0,\ x \geq 0\} \tag{3}$$

We note immediately that the piecewise-linear concave objective function of either (2) or (3) is in fact a natural residual for the general LCP and constitutes a local error bound for any LCP,

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and a global error bound for an LCP with positive definite M, or more generally for $M \in R_0$ [15, 6, 13, 10]. (The class R_0 is the class of matrices M for which 0 is the unique solution to the homogeneous LCP: $Mx \geq 0,\ x \geq 0,\ x'Mx = 0$.) It seems natural, then, to base an algorithm on attempting to drive this residual to zero, and this is what we plan to do in this paper.

We briefly outline now the contents. In Section 2 we state and establish finite termination of a stepless successive linearization algorithm (SLA) for finding a point satisfying the minimum principle necessary optimality condition for the problem of minimizing a concave function on a polyhedral set. In Section 3 we apply the algorithm to the general LCP via the formulation (3) and indicate its computational effectiveness by citing a specific instance [8] of successfully solving the knapsack feasibility problem as an LCP. Section 3 concludes the paper.

A word about our notation and background material. The feasible region of the LCP (1) is the set $\{x | Mx + q \geq 0,\ x \geq 0\}$. The scalar product of two vectors x and y in the n-dimensional real space will be denoted by $x'y$ in conformity with MATLAB [11] notation . For a linear program $\min_{x \in X} c'x$ with a vertex solution, the notation

$$\text{arg vertex} \min_{x \in X} \ c'x,$$

will denote the set of vertex solutions of the linear program. For $x \in R^n$, the norm $\|x\|$ will denote the 2-norm: $(x'x)^{\frac{1}{2}}$, while $\|x\|_1$ will denote the 1-norm: $\sum_{i=1}^{n} |x_i|$. The notation min $\{x, y\}$ applied to vectors x and y in R^n will denote a vector with components that are minima of corresponding components of x and y. For $x \in R^n$, $(x_+)_i = \max\ \{0, x_i\}$, $i = 1, \ldots, n$. For an $m \times n$ matrix A, A_i will denote the ith row of A. The identity matrix in a real space of arbitrary dimension will be denoted by I, while a column vector of ones of arbitrary dimension will be denoted by e. For a concave function $f : R^n \rightarrow R$ the supergradient $\partial f(x)$ of f at x is a vector in R^n satisfying

$$f(y) - f(x) \leq \partial f(x)(y - x) \tag{4}$$

for any $y \in R^n$. The set $D(f(x))$ of supergradients of f at the point x is nonemepty, convex, compact and reduces to the ordinary gradient $\nabla f(x)$, when f is differentiable at x [14, 16].

2 The Concave Minimization Algorithm

We consider in this section the following problem:

$$\min_{x \in X} f(x), \tag{5}$$

where $f : R^n \rightarrow R$ is a concave function on R^n and X is a polyhedral set in R^n that does not contain lines going to infinity in both directions. For such a problem, if f is bounded below on X, problem (5) has a vertex solution [16, Corollary 32.3.4]. We now state and establish finite termination of stepless successive linearization algorithm, which is an extension of an algorithm of [9] to nondifferentiable concave functions that is also very effective for the solution of machine learning problems [1, 2].

Algorithm 1 Successive Linearization Algorithm (SLA) *Start with a random $x^0 \in R^n$. Having x^i determine x^{i+1} as follows:*

$$x^{i+1} \in \text{arg vertex} \min_{x \in X} \ \partial f(x^i)(x - x^i) \tag{6}$$

Stop if $x^i \in X$ *and* $\partial f(x^i)(x^{i+1} - x^i) = 0$.

We will show that this algorithm terminates after a finite number of steps at a point satisfying a minimum principle necessary optimality condition. But first we will show that every local solution of a concave minimization problem satisfies such a minimum principle.

Lemma 2 Minimum Principle Necessary Optimality Condition *Let* $\bar{x}$ *be a local solution of* $\min_{x \in Y} f(x)$ *where* Y *is a convex set in* R^n *and* f *is a concave function on* R^n. *Then* $\bar{x}$ *satisfies the following minimum principle*

$$\partial f(\bar{x})(x - \bar{x}) \geq 0, \quad \forall x \in Y \tag{7}$$

Proof Let $\bar{x}$ a local solution, that is

$$f(\bar{x}) \leq f(y) \quad \forall y \in B(\bar{x}) \cap Y,$$

where $B(\bar{x})$ is some ball around $\bar{x}$. Then for any $x \in Y$ not in $B(\bar{x})$, it follows that

$$(1 - \lambda)\bar{x} + \lambda x \in B(\bar{x}) \text{ for some } \lambda \in (0, 1)$$

Hence

$$0 \leq f((1 - \lambda)\bar{x} + \lambda x) - f(\bar{x}) \leq \lambda \partial f(\bar{x})(x - \bar{x}),$$

where the the last inequality follows from (4). Noting that $\lambda > 0$, we immediately have the desired minimum principle (7). $\square$

We note that the minimum principle is usually given for *convex* minimization problems [14, Theorem 3, p. 203], [16, Theorem 27.4], and not for a concave minimization problem like the one under consideration here. Also, the proofs are completely different for the convex case, with the above proof being much simpler.

We are ready now to derive our finite termination result for the SLA 1.

Theorem 3 SLA Finite Termination Theorem *Let* f, *a concave function on* R^n, *be bounded below on* X. *The SLA generates a finite sequence of feasible iterates* $\{x^1, x^2, \ldots, x^{\bar{i}}\}$ *of strictly decreasing objective function values:* $f(x^1) > f(x^2) > \ldots > f(x^{\bar{i}})$, *such that* $x^{\bar{i}}$ *satisfies the minimum principle necessary optimality condition:*

$$\partial f(x^{\bar{i}})(x - x^{\bar{i}}) \geq 0, \quad \forall x \in X. \tag{8}$$

Proof We first show that SLA is well defined. By the concavity of f and its boundedness from below on X, we have that

$$-\infty < \inf_{x \in X} f(x) - f(x^i) \leq f(x) - f(x^i) \leq \partial f(x^i)(x - x^i), \quad \forall x \in X.$$

It follows for any $x^i \in R^n$, even for an infeasible x^i such as a possibly infeasible x^0, that $\partial f(x^i)(x - x^i)$ is bounded below on X. Hence the linear program (8) is solvable and has a vertex solution x^{i+1}. It follows for $i = 1, 2, \ldots,$ that

$$\forall x \in X: \ \partial f(x^i)(x - x^i) \geq \min_{x \in X} \partial f(x^i)(x - x^i) = \partial f(x^i)(x^{i+1} - x^i) \left\langle \begin{array}{ll} < 0 & \text{(a)} \\ = 0 & \text{(b)} \end{array} \right. \tag{9}$$

We note immediately that because $x^i \in X$ for $i = 1, 2, \ldots$, it follows that $\partial f(x^i)(x^{i+1} - x^i) \leq 0$. Hence only two cases , (a) or (b), can occur, as indicated above. When case (a) above occurs, the algorithm does not stop at iteration i, and we have from the concavity of f and the strict inequality of case (a) that:

$$f(x^{i+1}) \leq f(x^i) + \partial f(x^i)(x^{i+1} - x^i) < f(x^i).$$

Hence $f(x^{i+1}) < f(x^i)$, for $i = 1, 2, \ldots$. When case (b) occurs we then have that:

$$\forall x \in X: \;\; \partial f(x^i)(x - x^i) \geq 0, \tag{10}$$

and the algorithm terminates (provided $x^i \in X$, which may not be the case if $x^i = x^0 \notin X$), and we set $\bar{i} = i$. The point $x^{\bar{i}}$ thus satisfies the minimum principle necessary optimality conditions (8) with $x^{\bar{i}} = x^i$, and $x^{\bar{i}}$ may possibly be a global solution. Furthermore, since X has a finite number of vertices, $\{f(x^i)\}$ is strictly decreasing and $f(x)$ is bounded below on X, it follows that case (b) and hence (8) must occur after a finite number of steps. $\square$

We now turn to a specific application of the SLA 1 to the LCP.

3 The Concave Minimization Algorithm Applied to the LCP

We consider now the concave minimization formulation (3) of the LCP and apply SLA 1 to it. In order to do that we need to compute the supergradient of the objective function of (3) which is the following:

$$(\partial f(x)) = (\partial(e'\text{minimum}\{x, Mx+q\})) = \sum_{j=1}^{n} \left\langle \begin{array}{l} I_j \;\text{ if }\; x_j < M_j x + q_j \\ (1-\lambda) I_j + \lambda M_j \;\text{ if }\; x_j = M_j x + q_j,\; 0 \leq \lambda \leq 1 \\ M_j \;\text{ if }\; x_j > M_j x + q_j \end{array} \right\rangle \tag{11}$$

We can now apply the SLA 1 to the LCP by using the above supergradient for some fixed or varying λ. We summarize the algorithm and its finite termination to a stationary point as follows.

Algorithm 4 SLA for LCP *The SLA 1 applied to the LCP Problem (3) with*

$$f(x) = e'\text{minimum}\{x, Mx+q\},\; X = \{x \big| Mx + q \geq 0,\; x \geq 0\}$$

and supergradient defined by (11), terminates in a finite number of steps at a vertex $x^{\bar{i}}$ *of* X *satisfying the minimum principle necessary optimality condition (8).*

We note that the bilinear algorithm of [8] for solving the knapsack feasibility problem as an LCP can be interpreted as a special case of Algorithm 4 with a *fixed* $\lambda = 0$. That bilinear algorithm solved 80 consecutive instances of the knapsack LCP ranging in size between 10 and 3000 without failure. This is an indication that the proposed Algorithm 4 may be effective for classes of non-monotone LCPs.

4 Conclusion

We have established finite termination to a stationary point of a general stepless successive linearization algorithm applied to minimizing a nondifferentiable concave function on a polyhedral set and have applied it to a piecewise-linear concave formulation of the general LCP. The encouraging computational results of special cases of this algorithm applied to a knapsack LCP, as well to machine learning problems such as misclassification minimization [9], feature selection [1] and clustering [2], lead us to suggest that the proposed SLA 1 is a potential tool for solving important classes of difficult problems that are appropriately formulated as concave minimzation problems on polyhedral sets.

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