ON THE IDENTIFICATION OF ZERO VARIABLES IN AN INTERIOR-POINT FRAMEWORK*

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Abstract: We consider column sufficient linear complementarity problems and study the problem of identifying those variables that are zero at a solution. To this end we propose a new, computationally inexpensive technique that is based on growth functions. We analyze in detail the theoretical properties of the identification technique and test it numerically. The identification technique is particularly suited to interior-point methods but can be applied to a wider class of methods.

Key Words: Linear complementarity problem, column sufficient matrix, identification of zero variables, growth function, indicator function, interior-point method.

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1 Introduction

We consider the linear complementarity problem, LCP for short,

$$y = Mx + q, \qquad x \geq 0, \qquad y \geq 0, \qquad x^T y = 0,$$

where the matrix $M \in \mathbb{R}^{n \times n}$ and the vector $q \in \mathbb{R}^n$ are given. Throughout the paper we assume that

$$M \text{ is a column sufficient (CS) matrix}$$

(see [1, 2]), i.e., we assume that

$$x_i(Mx)_i \leq 0 \;\; \forall i \qquad \Longrightarrow \qquad x_i(Mx)_i = 0 \;\; \forall i.$$

We recall that positive semidefinite matrices and sufficient (or, equivalently, P_*-) matrices are column sufficient, so that the class of CS linear complementarity problems includes all the classes of LCPs for which interior-point methods have been extensively studied. We denote by $\mathcal{S}$ the solution set of LCP which is known to be convex for every q if and only if M is a CS matrix [1, Theorem 3.5.8]. We always assume that $\mathcal{S}$ is nonempty:

$$\mathcal{S} \neq \emptyset.$$

In this paper we are interested in techniques that identify the variables that are zero at a solution of an LCP. Obviously, the zero variables at a solution may be different from the zero variables at another solution. Therefore, in order to make more precise our aim, we define the following three index sets:

$$\begin{aligned} \mathcal{B} &:= \{i \,|\, x_i^* > 0, \text{ for at least one } (x^*, y^*) \in \mathcal{S}\}, \\ \mathcal{N} &:= \{i \,|\, y_i^* > 0, \text{ for at least one } (x^*, y^*) \in \mathcal{S}\}, \\ \mathcal{J} &:= \{i \,|\, x_i^* = y_i^* = 0, \text{ for all } (x^*, y^*) \in \mathcal{S}\}. \end{aligned}$$

The following proposition describes two properties of these index sets which are well-known in the case of a positive semidefinite matrix M.

Proposition 1.1 *(i) The index sets $\mathcal{B}$, $\mathcal{N}$ and $\mathcal{J}$ form a partition of $\{1, \ldots, n\}$.*
(ii) A point $z^* = (x^*, y^*) \in \mathcal{S}$ belongs to the relative interior $\mathrm{ri}\mathcal{S}$ of the solution set $\mathcal{S}$ if and only if

$$x_{\mathcal{B}}^* > 0, \quad x_{\mathcal{N}}^* = 0, \quad x_{\mathcal{J}}^* = 0, \qquad y_{\mathcal{B}}^* = 0, \quad y_{\mathcal{N}}^* > 0, \quad y_{\mathcal{J}}^* = 0. \tag{1}$$

Proof. (i) It is obvious that $\mathcal{B} \cup \mathcal{N} \cup \mathcal{J} = \{1, \ldots, n\}$. So we only have to show that $\mathcal{B}$, $\mathcal{N}$ and $\mathcal{J}$ are pairwise disjoint. In turn, since it is easy to see, by the very definition of these sets, that $\mathcal{B} \cap \mathcal{J} = \emptyset = \mathcal{N} \cap \mathcal{J}$, we only have to show that $\mathcal{B} \cap \mathcal{N} = \emptyset$. Suppose by contradiction that an index i belongs to both $\mathcal{B}$ and $\mathcal{N}$. Then there exist two points $(\bar{x}, \bar{y})$ and $(\hat{x}, \hat{y})$, both belonging to the solution set $\mathcal{S}$, such that $\bar{x}_i > 0$ and $\hat{y}_i > 0$. Consequently we have $\bar{y}_i = 0$ and $\hat{x}_i = 0$. Since M is column sufficient, $\mathcal{S}$ is convex. Therefore the point $(x(t), y(t)) = t(\bar{x}, \bar{y}) + (1-t)(\hat{x}, \hat{y})$ belongs to $\mathcal{S}$ for every $t \in (0, 1)$. But by the relations established above we have $x_i(t) > 0$ and $y_i(t) > 0$, thus contradicting the fact that $(x(t), y(t))$ belongs to $\mathcal{S}$.

(ii) The proof is identical to the one given in [3, Theorem 2.2] for monotone complementarity problems. A closer look at that proof shows that the monotonicity is used there only to establish the convexity of the solution set. Since the convexity of $\mathcal{S}$ holds under the assumption that M is column sufficient, the proof goes through. $\square$

Point (ii) of the above proposition shows that, in the relative interior of the set $\mathcal{S}$, the set of zero variables is invariant with respect to the solution. We recall that, under very mild assumptions, interior-point methods generate sequences of points such that every accumulation point is in the relative interior of $\mathcal{S}$ and so these solutions share the same zero-nonzero structure, see, e.g., [13, 14].

Our aim is to identify this structure or, equivalently, the sets $\mathcal{B}$, $\mathcal{N}$ and $\mathcal{J}$. The correct identification of these sets is important from both the theoretical and computational point of view. In fact, the knowledge of the zero-nonzero structure may allow, on the one hand, to easily recover an exact solution from the approximated one provided by an interior-point method and, on the other hand, to improve the efficiency of interior-point methods and column generation techniques [3].

The identification of the zero variables in interior-point methods for linear programs has been the subject of intense research in the last ten years, and we refer the reader to [3] for an exhaustive review. It is now accepted that the technique originally proposed by Tapia [16] for nonlinear programs enjoys the most interesting properties in the context of interior-point methods for linear programming [3].

This technique has also been extended to the case of linear complementarity problems [3, 4, 8, 13]. Then, however, a further difficulty can occur. In contrast to linear programs, where we always have $\mathcal{J} = \emptyset$, this is no longer true for LCP. Problems with $\mathcal{J} = \emptyset$ are called *nondegenerate*, while those for which $\mathcal{J} \neq \emptyset$ are termed *degenerate*. Degeneracy makes the identification of the sets $\mathcal{B}$, $\mathcal{N}$ and $\mathcal{J}$ more difficult [4, 8, 13].

In this paper we present a new technique for identifying the sets $\mathcal{B}$, $\mathcal{N}$ and $\mathcal{J}$. We show that, given a point $z = (x, y)$ belonging to a certain set $\mathcal{Z}_\varepsilon$, we are able to correctly identify $\mathcal{B}$, $\mathcal{N}$ and $\mathcal{J}$. The set $\mathcal{Z}_\varepsilon$ is defined in such a way that virtually all interior-point methods will generate, under standard assumptions, a sequence whose points eventually belong to this set, thus ensuring finite identification. We want to stress from the outset two peculiarities which, in our view, are significant. First, the class of problems we are able to deal with is considerably broader than the ones considered in previous works. Secondly, different to other works on the same subject, we do not make reference to a specific (although general) algorithmic scheme, so that the results obtained can be applied to a class of methods wider than the interior-point one.

The approach we use in this paper is reminiscent of the one proposed in [5] for general nonlinear programs. However there is a major difference: one of the key assumptions in [5] is that the solution of interest is an isolated solution. This assumption is not sensible in the LCP case and we therefore drop it by fully exploiting the structure of the problem. Furthermore, we are able to obtain particularly simple expressions for the growth functions (see Section 3) and convergence rates estimates (see Sections 4 and 5) that have no parallel in [5].

The paper is organized as follows. Below we introduce some further notation. In the next section we present the basic identification results of the paper. Sections 3 and 4 address some more technical points related to the identification technique. In Section 5 we specialize some of the results to an interior-point framework, while numerical experiments are reported in Section 6. In Section 7 we make some final comments.

Throughout the paper $\|\cdot\|$ denotes the Euclidean norm and $\mathrm{dist}(z|\mathcal{S}) := \inf_{w\in\mathcal{S}} \|w - z\|$ the Euclidean distance of the point z from the set $\mathcal{S}$. We define the set $\mathcal{Z}$ by

$$\mathcal{Z} := \{z = (x, y) \in \mathbb{R}^{n+n} \mid z \text{ satisfies Conditions (C1) – (C3)}\},$$

where

(C1) $x_{\mathcal{B}} \geq \delta, \quad y_{\mathcal{N}} \geq \delta,$

(C2) $\|z\| \leq C,$

(C3) $\|r\| \leq \eta\|Xy\|, \quad$ with $X := \mathrm{diag}(x_1, \ldots, x_n)$ and $r := r(z) := y - (Mx + q),$

and where $\delta > 0$, $C > 0$, and $\eta \geq 0$ are constants such that the interestion of $\mathcal{Z}$ and the solution set $\mathcal{S}$ is nonempty. Given a positive constant ε, we shall also consider the following set $\mathcal{Z}_\varepsilon$:

$$\mathcal{Z}_\varepsilon := \mathcal{Z} \cap \{z|\, \mathrm{dist}\,(z|\mathcal{S}) \leq \varepsilon\},$$

We note that under standard, mild assumptions the vast majority of existing interior-point methods for LCPs will produce a sequence of points which belongs to $\mathcal{Z}$ (for suitable δ, C and η) and to $\mathcal{Z}_\varepsilon$ (for any fixed positive ε) eventually. To see this, we may refer to the paper by Monteiro and Wright [13], where the authors introduce a general framework that covers a large number of interior-point methods for monotone LCPs. It can easily be seen that within this framework the Conditions (C1) – (C3) are satisfied. In particular, Condition (C2) is explicitly stated in property a) of that framework, whereas Condition (C3) can be directly obtained from property d). Moreover, Condition (C1) follows from [13, Lemma 2.2]. For LCPs with P_*-matrices Potra and Sheng [14] consider an infeasible interior-point method. Using Theorem 2.3, (2.5), and Theorem 4.1 in [14] one can verify that any sequence generated by the method from [14] eventually satisfies Conditions (C1) – (C3) for suitable δ, C and η.

2 Identification Results

This section contains the basic identification results of the paper. We shall show that, given any point z in $\mathcal{Z}_\varepsilon$, if ε is sufficiently small, we can correctly identify the sets $\mathcal{B}$, $\mathcal{N}$ and $\mathcal{J}$. To this end we need some preliminary results and definitions.

Proposition 2.1 *For any $z = (x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, it holds that*

$$|x_i| \leq \mathrm{dist}(z|\mathcal{S}) \quad \forall i \in \mathcal{N} \cup \mathcal{J}, \qquad |y_i| \leq \mathrm{dist}(z|\mathcal{S}) \quad \forall i \in \mathcal{B} \cup \mathcal{J}.$$

Proof. Let $z^\perp = (x^\perp, y^\perp)$ denote the orthogonal projection of $z = (x, y)$ onto $\mathcal{S}$ (we recall that $\mathcal{S}$ is a nonempty, closed and convex set, so that the orthogonal projection onto this set is uniquely defined). Since (1) holds for all $z^* \in \mathrm{ri}\mathcal{S}$ it follows that $x^*_{\mathcal{N}\cup\mathcal{J}} = 0$ and $y^*_{\mathcal{B}\cup\mathcal{J}} = 0$ for all $z^* \in \mathcal{S}$ so that

$$x^\perp_{\mathcal{N}\cup\mathcal{J}} = 0, \qquad y^\perp_{\mathcal{B}\cup\mathcal{J}} = 0.$$

Thus, we get for $i \in \mathcal{N} \cup \mathcal{J}$:

$$|x_i| = |x_i - 0| = |x_i - x_i^\perp| \leq \|x - x^\perp\| \leq \|z - z^\perp\| = \mathrm{dist}(z|\mathcal{S}).$$

Similar reasonings can be repeated for y_i, $i \in \mathcal{B} \cup \mathcal{J}$, and this completes the proof. □

The following two definitions are fundamental for our subsequent considerations.

Definition 2.2 *A function $\rho : \mathbb{R}^{n+n} \to [0, \infty)$ is called* growth function *on $\mathcal{Z}$ if there is a constant $c_1 \geq 1$ such that*

$$\frac{1}{c_1}\mathrm{dist}(z|\mathcal{S}) \leq \rho(z) \leq c_1 \mathrm{dist}(z|\mathcal{S}) \tag{2}$$

for all $z \in \mathcal{Z}$.

Note that Definition 2.2 implies that $\rho(z)$ is equal to 0 if and only if z is a solution of the linear complementarity problem.

Definition 2.3 *Let $\rho : \mathbb{R}^{n+n} \to [0, \infty)$ be a growth function on $\mathcal{Z}$ and $\alpha \in (0, 1)$ be fixed. Then the function $S : \mathbb{R} \times \mathcal{Z} \to \mathbb{R}$ defined by*

$$S(\xi, z; \alpha) := \begin{cases} \dfrac{\xi}{\xi - \rho(z)^\alpha} & \text{if } \xi \neq \rho(z)^\alpha \\ 0 & \text{otherwise} \end{cases}$$

is called an indicator function.

In the next section we show that growth functions and consequently indicator functions can easily be constructed. The following proposition justifies the name "indicator function".

Proposition 2.4 *For any $\alpha \in (0, 1)$ it holds that*

$$\lim_{\varepsilon \to 0, z \in \mathcal{Z}_\varepsilon} S(x_i, z; \alpha) = 1 \quad \forall i \in \mathcal{B}, \tag{3}$$

$$\lim_{\varepsilon \to 0, z \in \mathcal{Z}_\varepsilon} S(x_i, z; \alpha) = 0 \quad \forall i \in \mathcal{N} \cup \mathcal{J}, \tag{4}$$

$$\lim_{\varepsilon \to 0, z \in \mathcal{Z}_\varepsilon} S(y_i, z; \alpha) = 1 \quad \forall i \in \mathcal{N}, \tag{5}$$

$$\lim_{\varepsilon \to 0, z \in \mathcal{Z}_\varepsilon} S(y_i, z; \alpha) = 0 \quad \forall i \in \mathcal{B} \cup \mathcal{J}. \tag{6}$$

Proof. The fact that $0 \le \text{dist}(z|\mathcal{S}) \le \varepsilon \to 0$ and the right inequality in (2) imply $\rho(z) \to 0$. This fact and Condition (C1) yield (3).

Suppose now that $i \in \mathcal{N} \cup \mathcal{J}$. We need to consider only those (x_i, z) with $S(x_i, z; \alpha) \neq 0$. The very definition of the indicator function S then implies that $x_i \neq 0$. Using the left inequality of (2) and Proposition 2.1, we therefore have

$$\left|\frac{x_i - \rho(z)^\alpha}{x_i}\right| \ge \frac{\rho(z)^\alpha}{|x_i|} - 1 \ge \frac{\text{dist}(z|\mathcal{S})^\alpha}{c_1^\alpha |x_i|} - 1 \ge \frac{|x_i|^{\alpha-1}}{c_1^\alpha} - 1. \tag{7}$$

Proposition 2.1 and $\text{dist}(z|\mathcal{S}) \le \varepsilon \to 0$ imply $x_i \to 0$. Thus, by (7), it follows that

$$\lim_{\varepsilon \to 0, z \in \mathcal{Z}_\varepsilon} \frac{1}{|S(x_i, z, \alpha)|} = \lim_{\varepsilon \to 0, z \in \mathcal{Z}_\varepsilon} \left|\frac{x_i - \rho(z)^\alpha}{x_i}\right| = \infty,$$

i.e., (4) is valid.

The limits (5) and (6) can be proved similarly. □

The above result suggests to introduce the following approximations to the sets $\mathcal{B}$, $\mathcal{N}$ and $\mathcal{J}$. Let $\theta \in (0, 1/2)$ and $\alpha \in (0,1)$ be fixed and ρ be a given growth function on $\mathcal{Z}$; define

$$\begin{aligned}
\mathcal{B}(z;\alpha) &:= \{i|\, S(x_i, z; \alpha) \ge 1 - \theta\}, \\
\mathcal{N}(z;\alpha) &:= \{i|\, S(y_i, z; \alpha) \ge 1 - \theta\}, \\
\mathcal{J}(z;\alpha) &:= \{i|\, \max\{S(x_i, z; \alpha), S(y_i, z; \alpha)\} \le \theta\}.
\end{aligned}$$

Note that these three sets are pairwise disjoint, but they do not necessarily form a partition of $\{1, \ldots, n\}$. The following result is the principal result of this section and shows that the sets just defined are indeed reasonable estimates of the sets $\mathcal{B}$, $\mathcal{N}$ and $\mathcal{J}$.

Theorem 2.5 *Let $\alpha \in (0,1)$ be given. Then there is an $\varepsilon > 0$ such that*

$$\mathcal{B}(z;\alpha) = \mathcal{B}, \qquad \mathcal{N}(z;\alpha) = \mathcal{N}, \qquad \mathcal{J}(z;\alpha) = \mathcal{J} \tag{8}$$

for all $z \in \mathcal{Z}_\varepsilon$.

Proof. Assume the contrary. Then sequences $\{\varepsilon^k\} \to 0$ and $\{z^k\}$ exist such that, for every k, $z^k \in \mathcal{Z}_{\varepsilon^k}$ and at least one of the equalities in (8) is violated.

Since ε^k converges to 0, we have that $\text{dist}(z^k|\mathcal{S})$ also converges to 0, so that (3) – (6) hold. This obviously implies that all the equalities in (8) hold eventually. Therefore we got a contradiction and the proof is complete. □

Remark 2.6 Using the indicator function and their properties we can easily define different approximations to the sets $\mathcal{B}$, $\mathcal{N}$ and $\mathcal{J}$. For example, in Section 6 we shall use the following approximations in the numerical tests:

$$\begin{aligned}
\mathcal{B}'(z;\alpha) &:= \{i|\, \min\{S(x_i, z; \alpha), 1 - S(y_i, z; \alpha)\} \ge 1 - \theta\}, \\
\mathcal{N}'(z;\alpha) &:= \{i|\, \min\{S(y_i, z; \alpha), 1 - S(x_i, z; \alpha)\} \ge 1 - \theta\}, \\
\mathcal{J}'(z;\alpha) &:= \mathcal{J}(z;\alpha).
\end{aligned}$$

It is easy to see that these approximations enjoy the same properties established in Theorem 2.5 and that $\mathcal{B}'(z;\alpha) \subseteq \mathcal{B}(z;\alpha)$ and $\mathcal{N}'(z;\alpha) \subseteq \mathcal{N}(z;\alpha)$ so that $\mathcal{B}'(z;\alpha)$ and $\mathcal{N}'(z;\alpha)$ may be seen as more restrictive versions of the approximations $\mathcal{B}(z;\alpha)$ and $\mathcal{N}(z;\alpha)$.

3 Growth Functions

We saw in the previous section that a key role in the identification of the zero-nonzero pattern of the solutions is played by growth functions. In this section we show that growth functions can be defined easily.

Before presenting a first example of a growth function, we need some preliminary results. Consider the projection of the solution set $\mathcal{S}$ on the space of x-variables and indicate it by $\mathcal{S}_x$:

$$\mathcal{S}_x := \{x \in \mathrm{I\!R}^n \,|\, \exists y \in \mathrm{I\!R}^n : (x, y) \in \mathcal{S}\}. \tag{9}$$

Since, in view of our general assumptions, the solution set $\mathcal{S}$ is nonempty, closed and convex, also $\mathcal{S}_x$ is nonempty, closed and convex. The following lemma gives an error bound result for the set

$$\mathcal{Z}_x := \{x \in \mathrm{I\!R}^n \,|\, \exists y \in \mathrm{I\!R}^n : (x, y) \in \mathcal{Z}\},$$

which, by Condition (C2), is bounded.

Lemma 3.1 *There is a constant $c_2 > 0$ such that*

$$\operatorname{dist}(x|\mathcal{S}_x) \le c_2 \|\min\{x, Mx + q\}\|$$

for all $x \in \mathcal{Z}_x$.

Proof. It can be easily derived from [10] that, given a point $\bar{x} \in \mathcal{S}_x$, there exist a constant $\kappa_1 > 0$ and a neighborhood Ω of $\bar{x}$ such that

$$\operatorname{dist}(x|\mathcal{S}_x) \le \kappa_1 \|\min\{x, Mx + q\}\|, \qquad \forall x \in \Omega. \tag{10}$$

Suppose now that the lemma is false. Then a sequence $\{x^k\}$ contained in $\mathcal{Z}_x$ exists such that

$$\operatorname{dist}(x^k|\mathcal{S}_x) > k \|\min\{x^k, Mx^k + q\}\| \tag{11}$$

for all k. Since $\mathcal{Z}_x$ is bounded, we can assume without loss of generality that $\{x^k\}$ converges to a point $\bar{x}$. It is also easy to see that $\bar{x}$ belongs to $\mathcal{S}_x$ for, if this were not true, (11) would imply $\operatorname{dist}(x^k, \mathcal{S}_x) \to \infty$, which, in view of the boundedness of $\mathcal{Z}_x$, is impossible. But if $\bar{x}$ belongs to $\mathcal{S}_x$, we have that eventually, (11) contradicts (10). $\square$

Using Lemma 3.1 we can now give an error bound result for the solution set $\mathcal{S}$.

Lemma 3.2 *There is a constant $c_3 > 0$ such that*

$$\operatorname{dist}(z|\mathcal{S}) \le c_3 \left(\|\min\{x, y\}\| + \eta \|Xy\|\right) \tag{12}$$

for all $z \in \mathcal{Z}$, where $\eta \ge 0$ denotes the constant from Condition (C3).

Proof. Let $z \in \mathcal{Z}$ with $z = (x, y)$ be given. Since, as noted before, $\mathcal{S}_x$ is nonempty, closed and convex, there exists an orthogonal projection $x^\perp$ of $x \in \mathbb{R}^n$ on the set $\mathcal{S}_x$. By the definition of $\mathcal{S}_x$, there is a vector $y^\perp$ such that $z^\perp = (x^\perp, y^\perp) \in \mathcal{S}$. Thus, we get

$$\begin{array}{rcl} \operatorname{dist}(z|\mathcal{S}) & \leq & \|z - z^\perp\| \\ & \leq & \|x - x^\perp\| + \|y - y^\perp\| \\ & = & \|x - x^\perp\| + \|M(x - x^\perp) + r\| \\ & \leq & (1 + \|M\|)\|x - x^\perp\| + \|r\|. \end{array} \tag{13}$$

Using Lemma 3.1, we have

$$\operatorname{dist}(x|\mathcal{S}_x) \leq c_2\|\min\{x, Mx + q\}\| = c_2\|\min\{x, y - r\}\|, \tag{14}$$

where the equality follows directly from the definition of the vector r in Condition (C3).

Now, taking into account the easily verified relation

$$|\min\{a, b + c\}| \leq |\min\{a, b\}| + |c|, \qquad \forall a,\, b,\, c \in \mathbb{R},$$

and the fact that all norms are equivalent in $\mathbb{R}^n$, it follows that there is a constant $\kappa_2 > 0$ such that

$$\|\min\{x, y - r\}\| \leq \kappa_2 \left(\|\min\{x, y\}\| + \|r\|\right). \tag{15}$$

Combining the inequalities (13)–(15) and using (C3), we therefore get

$$\begin{array}{rcl} \operatorname{dist}(z|\mathcal{S}) & \leq & (1 + \|M\|)\operatorname{dist}(x|\mathcal{S}_x) + \|r\| \\ & \leq & (1 + \|M\|)c_2\|\min\{x, y - r\}\| + \|r\| \\ & \leq & (1 + \|M\|)\kappa_2 c_2\|\min\{x, y\}\| + (1 + \|M\|)\kappa_2 c_2\eta\|Xy\| + \eta\|Xy\| \\ & \leq & c_3(\|\min\{x, y\}\| + \eta\|Xy\|), \end{array}$$

where

$$c_3 := (1 + \|M\|)\kappa_2 c_2 + 1.$$

This is exactly the desired inequality. $\square$

We are now in the position to present a first example of a growth function.

Proposition 3.3 *The function $\rho_1 : \mathbb{R}^{n+n} \to [0, \infty)$, defined by*

$$\rho_1(z) := \|\min\{x, y\}\|,$$

is a growth function on $\mathcal{Z}$.

Proof. Taking into account Condition (C2) and that $|ab| = |\max\{a, b\}||\min\{a, b\}|$ is valid for arbitrary $a, b \in \mathbb{R}$, we obtain

$$\|Xy\| = \sqrt{\sum_{i=1}^{n}(x_i y_i)^2} \leq \sum_{i=1}^{n}|x_i y_i| \leq C\sum_{i=1}^{n}|\min\{x_i, y_i\}| \leq C\sqrt{n}\rho_1(z), \tag{16}$$

where $C > 0$ denotes the constant from Condition (C2). Using Lemma 3.2 we therefore have, for all $z \in \mathcal{Z}$,

$$\mathrm{dist}(z|\mathcal{S}) \leq c_3\rho_1(z) + c_3\eta\|Xy\| \leq \kappa_3\rho_1(z),$$

where $\kappa_3 = c_3(1+C\eta\sqrt{n})$. On the other hand, the function ρ_1 is globally Lipschitz continuous on $\mathbb{R}^{n+n}$, see [9]; let L be its Lipschitz constant. Then, denoting by $z^\perp$ the orthogonal projection of z onto $\mathcal{S}$, we get

$$\rho_1(z) = |\rho_1(z) - \rho_1(z^\perp)| \leq L\|z - z^\perp\| = L\mathrm{dist}(z|\mathcal{S})$$

for each $z \in \mathcal{Z}$. Hence, ρ_1 satisfies Definition 2.2 with $c_1 := \max\{\kappa_3, L\}$. □

Using the previous proposition it is now easy to build other growth functions. In the next corollary we give two more examples.

Corollary 3.4 *The functions $\rho_2, \rho_3 : \mathbb{R}^{n+n} \to [0, \infty)$ defined by*

$$\rho_2(z) := \|(\sqrt{x_1^2 + y_1^2} - x_1 - y_1, \dots, \sqrt{x_n^2 + y_n^2} - x_n - y_n)\|$$

and

$$\rho_3(z) := \|\min\{x, y\}\| + \|Xy\|$$

are growth functions on $\mathcal{Z}$.

Proof. It is known, see [17], that a positive constant κ_4 exists such that

$$\frac{1}{\kappa_4}\rho_1(z) \leq \rho_2(z) \leq \kappa_4\rho_1(z), \qquad \forall z \in \mathbb{R}^{2n}$$

From these relations and from Proposition 3.3 it then easily follows that ρ_2 is a growth function on $\mathcal{Z}$.

We next examine ρ_3. Because of Proposition 3.3 and (16), it follows immediately from the definitions of ρ_1 and ρ_3 that

$$\frac{1}{c_1}\mathrm{dist}(z|\mathcal{S}) \leq \rho_1(z) \leq \rho_3(z) \leq (1 + C\sqrt{n})\rho_1(z) \leq c_1(1 + C\sqrt{n})\mathrm{dist}(z|\mathcal{S})$$

for all $z \in \mathcal{Z}$, i.e., ρ_3 is a growth function. □

4 Rates of Convergence

The main point to consider when assessing the quality of estimates $\mathcal{B}(z, \alpha)$, $\mathcal{N}(z, \alpha)$ and $\mathcal{J}(z, \alpha)$ is: How large is the region where these estimates coincide with the sets they approximate? Unfortunately, it seems difficult to give theoretical results in this direction, and the only way we know to treat this point is through numerical experiments. However, in an effort to get some theoretical insight into this problem, some researchers turned to the study of the convergence rates of the indicator function values when z tends to the solution set $\mathcal{S}$. In this section we consider this issue. On the other hand, we think that the importance of these results should not be overestimated since the connection between convergence rates and the wideness of the region of correct identification is, from the theoretical point of view, loose.

We first state a technical lemma.

Lemma 4.1 *The inequality*

$$0 \leq \frac{\xi}{\xi - r} - 1 \leq \frac{4r}{\xi}$$

holds for all $\xi, r \in \mathbb{R}$ *with* $\xi > 0$ *and* $0 \leq r \leq 0.75\xi$.

Proof. The left inequality is obvious. On the other hand, the inequality on the right-hand side is equivalent to

$$\frac{r}{\xi - r} \leq \frac{4r}{\xi}$$

which, in turn, is equivalent to

$$r\xi \leq 4r(\xi - r) = 4r\xi - 4r^2$$

since $\xi > 0$ and $\xi - r > 0$. Now, this inequality is satisfied if and only if

$$0 \leq 3\xi - 4r,$$

and this is true because $r \leq 0.75\xi$ by assumption. □

The following result relates the convergence rate of the indicator functions to the convergence rate of the distance of the point z to the solution set $\mathcal{S}$.

Theorem 4.2 *Let* $\alpha \in (0, 1)$ *be given. Then, for* $z \in \mathcal{Z}$ *with* $\operatorname{dist}(z|\mathcal{S}) \to 0$*, it holds*

$$|S(x_i, z; \alpha) - 1| = O(\operatorname{dist}(z|\mathcal{S})^\alpha) \qquad \forall i \in \mathcal{B}, \tag{17}$$

$$|S(y_i, z; \alpha) - 1| = O(\operatorname{dist}(z|\mathcal{S})^\alpha) \qquad \forall i \in \mathcal{N}, \tag{18}$$

$$|\max\{S(x_i, z; \alpha), S(y_i, z; \alpha)\}| = O(\operatorname{dist}(z|\mathcal{S})^{1-\alpha}) \qquad \forall i \in \mathcal{J}. \tag{19}$$

Proof. We prove (17) by applying Lemma 4.1 with $\xi := x_i$ and $r := \rho(z)^\alpha$. So let $i \in \mathcal{B}$ be an arbitrary but fixed index. Since we need to consider only $z \in \mathcal{Z}_\varepsilon$ with $\operatorname{dist}(z|\mathcal{S}) \leq \varepsilon$ sufficiently small, $r = \rho(z)^\alpha \leq 0.75 x_i = 0.75\xi$ follows for these z because of Condition (C1) and (2). Moreover, $\xi = x_i > 0$ is obvious. Therefore, Lemma 4.1 can be applied and yields, having Condition (C1) and (2) in mind,

$$|S(x_i, z; \alpha) - 1| = \frac{x_i}{x_i - \rho(z)^\alpha} - 1 \leq \frac{4}{x_i}\rho(z)^\alpha \leq \frac{4}{\delta} c_1^\alpha \operatorname{dist}(z|\mathcal{S})^\alpha \quad \forall i \in \mathcal{B}. \tag{20}$$

The proof of (18) is similar and we omit it.

Now, consider an arbitrary but fixed $i \in \mathcal{J}$. To prove (19) we first show that $|S(x_i, z; \alpha)| = O(\mathrm{dist}(z|\mathcal{S})^{1-\alpha})$. Since only those $z \in \mathcal{Z}_\varepsilon$ with $S(x_i, z; \alpha) \neq 0$ need to be considered, the definition of the indicator function S immediately implies that $x_i \neq 0$. Since $i \in \mathcal{J}$, this means that z is not a solution of the linear complementarity problem so that $\rho(z) > 0$. Using

$$\rho(z) \leq c_1 \mathrm{dist}(z|\mathcal{S}) \leq c_1 \varepsilon$$

for all $z \in \mathcal{Z}_\varepsilon$, we therefore obtain the existence of a sufficiently small $\varepsilon > 0$ such that

$$c_1 \rho(z) = [c_1 \rho(z)^{1-\alpha}] \rho(z)^\alpha < \rho(z)^\alpha$$

holds for all $z \in \mathcal{Z}_\varepsilon$. Now Proposition 2.1 and (2) imply that

$$|x_i| \leq \mathrm{dist}(z|\mathcal{S}) \leq c_1 \rho(z) < \rho(z)^\alpha$$

for all $z \in \mathcal{Z}_\varepsilon$ with $z = (x, y)$. Thus, we can introduce $a(z) := \rho(z)^\alpha / |x_i|$ and observe that, by Proposition 2.1, (2), and $\rho(z) > 0$,

$$0 < \frac{1}{a(z)} \leq c_1^\alpha \mathrm{dist}(z|\mathcal{S})^{1-\alpha}. \tag{21}$$

This yields $a(z) \to \infty$ for $\mathrm{dist}(z|\mathcal{S}) \to 0$. Therefore, we have, for $z \in \mathcal{Z}_\varepsilon$ with ε sufficiently small,

$$|S(x_i, z; \alpha)| = \left| \frac{x_i}{x_i - \rho(z)^\alpha} \right| = \frac{1}{|1 - \rho(z)^\alpha / x_i|} \leq \frac{1}{|a(z)| - 1} \leq \frac{2}{a(z)}.$$

Together with (21) this gives

$$|S(x_i, z; \alpha)| = O(\mathrm{dist}(z|\mathcal{S})^{1-\alpha}).$$

The same result can be shown for $|S(y_i, z; \alpha)|$ in a similar way so that (19) follows. $\square$

5 Rates of Convergence and Complementarity Gap

The result in the previous section is geometrically very appealing, since it relates the convergence rates of the indicator functions to the Euclidean distance to the solution set. However, in connection with interior-point methods, it is also important to relate this distance to the normalized complementarity gap

$$\mu := \mu(z) := \frac{x^T y}{n}.$$

In fact, in interior-point methods a convergence rate is often established for μ (and not for the distance), see, for example, the recent books [15, 18, 19] for a general background on interior-point methods.

Instead of the set $\mathcal{Z}$ we will now make use of its nonnegative part

$$\mathcal{Z}^+ := \{z \in \mathcal{Z} \mid z \geq 0\}.$$

Note that virtually every interior-point method will generate sequences $\{z^k\}$ belonging to $\mathcal{Z}^+$ eventually.

Before giving the main result of this section we relate the distance $\mathrm{dist}(z|\mathcal{S})$ to the complementarity gap.

Proposition 5.1 *If $\mathcal{J} \neq \emptyset$, there is a constant $c_4 > 0$ such that*

$$\operatorname{dist}(z|\mathcal{S}) \leq c_4\sqrt{\mu} \tag{22}$$

for all $z \in \mathcal{Z}^+$. If, instead, $\mathcal{J} = \emptyset$, then there is a constant $c_5 > 0$ such that

$$\operatorname{dist}(z|\mathcal{S}) \leq c_5\mu \tag{23}$$

for all $z \in \mathcal{Z}^+$ sufficiently close to $\mathcal{S}$.

Proof. From Lemma 3.2, we have

$$\operatorname{dist}(z|\mathcal{S}) \leq c_3\left(\|\min\{x,y\}\| + n\eta\mu\right). \tag{24}$$

Since $\min\{a,b\} \leq \sqrt{ab}$ is valid for arbitrary $a, b \geq 0$ and since $z \geq 0$, we obtain that

$$\|\min\{x,y\}\|^2 = \sum_{i=1}^{n} \min^2\{x_i, y_i\} \leq \sum_{i=1}^{n} x_i y_i = x^T y = n\mu.$$

This and (24) gives

$$\operatorname{dist}(z|\mathcal{S}) \leq c_3\left(\sqrt{n} + n\eta\sqrt{\mu}\right)\sqrt{\mu}.$$

In view of Condition (C2), there is a constant $\kappa_5 > 0$ such that

$$\sqrt{\mu} = \sqrt{\frac{x^T y}{n}} \leq \kappa_5$$

for all $z = (x,y) \in \mathcal{Z}^+$. Hence it follows that

$$\operatorname{dist}(z|\mathcal{S}) \leq c_4\sqrt{\mu}$$

for $c_4 := c_3(\sqrt{n} + n\eta\kappa_5)$.

If $\mathcal{J} = \emptyset$, we have $\mathcal{B} \cup \mathcal{N} = \{1, \ldots, n\}$ by Proposition 1.1. Thus, Condition (C1) gives

$$\min\{x_i, y_i\} \leq \frac{x_i y_i}{\delta}$$

for every i and for every $z \in \mathcal{Z}^+$ sufficiently close to $\mathcal{S}$. Hence, we get from Condition (C1):

$$\|\min\{x,y\}\| \leq \frac{1}{\delta}\left(\sum_{i=1}^{n} x_i^2 y_i^2)\right)^{1/2} \leq \frac{1}{\delta}\sum_{i=1}^{n} x_i y_i = \frac{n}{\delta}\mu.$$

Inequality (23) now follows from (24) by taking $c_5 := c_3(n/\delta + n\eta)$. $\square$

If the matrix M is assumed to be positive semidefinite, Proposition 5.1 can be derived from known error bound results. We refer the reader to [12, 13] for the case $\mathcal{J} \neq \emptyset$ and to [11] for $\mathcal{J} = \emptyset$. Here we proved Proposition 5.1 under the mere conditions that $z \in \mathcal{Z}^+$ and that M is column sufficient.

In the next theorem we give convergence rates with respect to μ. These convergence rates easily follow from Theorem 4.2 and Proposition 5.1.

Theorem 5.2 *Let $\alpha \in (0,1)$ be given. If $\mathcal{J} \neq \emptyset$ then, for $z \in \mathcal{Z}^+$ and $\mu \to 0$, it holds*

$$\begin{aligned} |S(x_i, z; \alpha) - 1| &= O(\mu^{\alpha/2}) \qquad \forall i \in \mathcal{B}, \\ |S(y_i, z; \alpha) - 1| &= O(\mu^{\alpha/2}) \qquad \forall i \in \mathcal{N}, \\ |\max\{S(x_i, z; \alpha), S(y_i, z; \alpha)\}| &= O(\mu^{(1-\alpha)/2}) \qquad \forall i \in \mathcal{J}. \end{aligned}$$

If, instead, $\mathcal{J} = \emptyset$ then, for $z \in \mathcal{Z}^+$ and $\mu \to 0$, it holds

$$\begin{aligned} |S(x_i, z; \alpha) - 1| &= O(\mu^{\alpha}) \qquad \forall i \in \mathcal{B}, \\ |S(y_i, z; \alpha) - 1| &= O(\mu^{\alpha}) \qquad \forall i \in \mathcal{N}. \end{aligned}$$

Theorem 4.2 and Theorem 5.2 clearly show that the convergence rate of the indicator functions depend on α. In general if we want to maximize the slower convergence rate, the best value for α is 0.5. On problems which are known to be nondegenerate, for example in the linear programming case, a value of α close to 1 may be preferred instead. The different way in which α influences the convergence rate of nondegenerate and degenerate indices also suggests the idea of using two different values of α: a value close to 1 in the definition of $\mathcal{B}(z;\alpha)$ and $\mathcal{N}(z;\alpha)$, and a value close to 0 in the definition of $\mathcal{J}(z;\beta)$ (where we used the symbol β to point out that this value is different from the one used in the approximation of nondegenerate indices). It is not difficult to see that all the results we proved go through after this minor modification. However, in this case the sets $\mathcal{B}(z;\alpha)$, $\mathcal{N}(z;\alpha)$ and $\mathcal{J}(z;\beta)$ need not be everywhere pairwise disjoint, even if this will always be the case eventually, and this may require the definition of additional rules to decide to which set to assign an index which belongs to more than one set among $\mathcal{B}(z;\alpha)$, $\mathcal{N}(z;\alpha)$ and $\mathcal{J}(z;\beta)$.

6 Numerical Results

In order to get a feeling for the practical results that can be obtained with the new identification technique, in this section we present a summary of the results of an extensive numerical testing [6]. We report the results obtained by using:

(i) the Tapia indicator [3, 8, 13], probably the best indicator available to date for linear programs [3],

(ii) the new indicator, and

(iii) the intersection indicator, that is a combination of Tapia indicator and the new indicator.

The Tapia indicator and its characteristics are studied in detail in references [3, 13, 8]. Here we only recall some essential facts:

(a) The Tapia indicator can be applied only to a specific (although broad) class of interior-point methods for linear complementarity problems.

(b) Given a sequence of points $\{z^k\}$ generated by a suitable interior-point method and converging to the solution set of LCP, quantities T_x^k and T_y^k are associated with each $z^k = (x^k, y^k)$ so that, under assumptions which are similar, but stronger than conditions (C1)–(C3) used in this paper,

$$\lim_{k\to\infty} T_x^k = \begin{cases} 1 & \text{if } i \in \mathcal{B}, \\ 0.5 & \text{if } i \in \mathcal{J}, \\ 0 & \text{if } i \in \mathcal{N}. \end{cases} \qquad \lim_{k\to\infty} T_y^k = \begin{cases} 0 & \text{if } i \in \mathcal{B}, \\ 0.5 & \text{if } i \in \mathcal{J}, \\ 1 & \text{if } i \in \mathcal{N}. \end{cases}$$

We tested the three identification strategies on the `netlib` collection of linear programming problems. Although our identification technique can be applied to a much broader class of problems, we believe that linear programming represents the major field of application of the techniques described in this paper; furthermore, no collection of (column sufficient) linear complementarity test problems comparable to `netlib` exists to date. Therefore we decided, in this first stage of our numerical experience, to deal only with LPs. We stress, however, that these tests cover only a special case of the theory developed in the previous sections. In fact, the Goldman-Tucker theorem, see [7] or [18] for a more recent reference, shows that any linear program is nondegenerate, so that in our tests we never encounter the case $\mathcal{J} \neq \emptyset$.

For each problem we used the LIPSOL program by Zhang [20, 21] to generate a sequence of points converging to the solution set of the linear program. LIPSOL is a MATLAB/FORTRAN implementation of a predictor-corrector infeasible interior-point method. We added some lines in this code in order to calculate, at each iteration, approximations of the index sets $\mathcal{B}$ and $\mathcal{N}$ (recall that $\mathcal{J} = \emptyset$). More precisely, within each iteration, we calculate the values $S(x_i^k, z^k; \alpha)$ and $S(y_i^k, z^k; \alpha)$ after each corrector step and T_x^k and T_y^k after each predictor step. Based on these values, we approximate the index sets $\mathcal{B}$ and $\mathcal{N}$ at iteration k as follows:

(i) For the Tapia indicator we set:

$$\begin{aligned} \mathcal{B}_T^k &:= \{i \mid \min\{T_x^k, 1 - T_y^k\} \geq 1 - \theta\}, \\ \mathcal{N}_T^k &:= \{i \mid \min\{T_y^k, 1 - T_x^k\} \geq 1 - \theta\}. \end{aligned}$$

(ii) For the new indicator we set, in a similar way:

$$\begin{aligned} \mathcal{B}_S^k &:= \{i \mid \min\{S(x_i^k, z^k; \alpha), 1 - S(y_i^k, z^k; \alpha)\} \geq 1 - \theta\}, \\ \mathcal{N}_S^k &:= \{i \mid \min\{S(y_i^k, z^k; \alpha), 1 - S(x_i^k, z^k; \alpha)\} \geq 1 - \theta\}. \end{aligned}$$

(iii) Finally, for the intersection indicator, we calculate approximations $\mathcal{B}_{ST}^k$ and $\mathcal{N}_{ST}^k$ by intersecting the previous estimates:

$$\begin{aligned} \mathcal{B}_{ST}^k &:= \mathcal{B}_S^k \cap \mathcal{B}_T^k, \\ \mathcal{N}_{ST}^k &:= \mathcal{N}_S^k \cap \mathcal{N}_T^k. \end{aligned}$$

The rationale behind this last estimate is simply that our new indicator and the Tapia indicator are based on a totally different approach, so that if an index is estimated to be active (or non active) by both indicators then, and only then, we expect this prediction to be true. Note that the use of two indicators to confirm the information obtained from each one of them is also advocated in [3].

For all test runs we chose $\theta = 0.1$. Moreover, we set $\alpha = 0.5$ at the beginning of each test run and updated α after each step by

$$\alpha = \max\{\alpha, 1 - 100 * \texttt{trerror}\}$$

where `trerror` denotes a certain residual calculated within the LIPSOL program which, basically, measures the violation of the optimality conditions at the current iterate.

The first problem we have to tackle when analyzing the results is how to assess the quality of an indicator. While it is intuitively clear that an indicator is good if it can accurately estimate, at an early stage, the sets $\mathcal{B}$ and $\mathcal{N}$, it is not entirely clear the exact way we should measure this accuracy. In our experiments we chose to consider the following three quality indices. For simplicity we describe them making reference to the new estimates $\mathcal{B}_S^k$ and $\mathcal{N}_S^k$ only, but it is obvious that analogous considerations can be made with reference to the Tapia indicator and to the intersection indicator.

`Percentage of misclassified indices at iteration` k. At each iteration a variable, x_i for example, can either be classified as active ($i \in \mathcal{N}_S^k$), or non-active ($i \in \mathcal{B}_S^k$) or can be not classified at all ($i \notin \mathcal{N}_S^k$ and $i \notin \mathcal{B}_S^k$). The percentage of misclassified indices at iteration k is the number of indices estimated to belong to $\mathcal{B}$ ($\mathcal{N}$) at that iteration and that instead, at a solution belonging to the relative interior of the solution set, belong to $\mathcal{N}$ ($\mathcal{B}$). In formulas this corresponds to

$$100 \frac{\left|\mathcal{B}_S^k \setminus \mathcal{B}\right| + \left|\mathcal{N}_S^k \setminus \mathcal{N}\right|}{n}.$$

`Percentage of correctly classified indices at iteration` k. This is easily understood to be

$$100 \frac{\left|\mathcal{B}_S^k \cap \mathcal{B}\right| + \left|\mathcal{N}_S^k \cap \mathcal{N}\right|}{n}.$$

`Percentage of globally correctly classified indices at iteration` k. We say that a certain index is globally correctly identified at iteration k if its identification status is correct at iteration k and does not change from that iteration on.

Roughly speaking, the first quality index described above measures the excess of $\mathcal{B}_S^k$ and $\mathcal{N}_S^k$ over $\mathcal{B}$ and $\mathcal{N}$ respectively; while the second index measures the the excess of $\mathcal{B}$ and $\mathcal{N}$ over $\mathcal{B}_S^k$ and $\mathcal{N}_S^k$ respectively. All the indices are correctly classified at iteration k if the percentage of misclassified indices is 0 and that of correctly classified ones is 100. However, neither of the two quality indices alone allows us to assess the quality of the current guessing. The third quality index is similar to the second one with a greater emphasis on stability of the indicators, though. According to one's purposes one of the three quality indices above may be more important than the others, and other indices may be of interest too. However we think that these three quality indices, considered together, give a fairly reasonable picture of the behavior of the indicators.

There is another difficulty we must mention. The evaluation of the above quality indices assumes the knowledge of $\mathcal{B}$ and $\mathcal{N}$, but this is not the case, in general, for the `netlib` problems we used. Therefore we assumed that if in the final iteration the estimates obtained using the new indicator and the Tapia indicator coincide, i.e., if at the last iteration $\mathcal{B}_S^k = \mathcal{B}_T^k$ and $\mathcal{N}_S^k = \mathcal{N}_T^k$ holds, then these estimates coincide with $\mathcal{B}$ and $\mathcal{N}$. We run LIPSOL on all the problems using the default parameters, but it turned out that on a considerably high percentage of problems the new indicator and the Tapia one do not coincide at the last iteration. Hence we changed the main stopping criterion (TOL) of LIPSOL from 10^{-8} to 10^{-11}. The satisfaction of this more stringent termination criterion usually required only one or two additional iterations and increased the number of problems on which the two indicators coincide at the last iteration.

Unfortunately, it is not always possible to reach this higher accuracy and we were therefore forced to consider only the 73 problems that were successfully solved with TOL=10^{-11}. For 9 of these 73 problems we do not have coincidence of the indications obtained by the new and the Tapia indicator. Since the resulting set of 64 test problems appears to be significant we did not adopt further expedients to enlarge this set of test problems. In the next three subsections we summarize the behaviour of the indicators on the test problem set. Because of lack of space, it is impossible to report here the complete numerical results. We tried to give a fair representation of these results by reporting some summary tables that highlight the main features of the indicators. However, it should always be kept in mind that our comments are based on the complete set of numerical results. The interested reader can find the complete and detailed numerical results in the companion report [6].

6.1 Misclassified Indices

We recall that the percentage of misclassified indices appears to be particularly important in those cases in which one wants to reduce the dimension of the problem by fixing variables to 0. In this case a high number of misclassified indices can adversely affect the efficiency of the procedure (see [3]). More in general, we tend to view this index as an important one because it tells us how much we can trust the guessing. It is useless to have a high percentage of correctly identified indices (something assessed by the indices analyzed in the next two sections) if these indices are mixed with too many misclassified ones. We also recall that the misclassified indices should not be confused with the unclassified ones.

We summarize the results in two tables. In Table 6.1 we report the number of test examples for which we have less than 1% of misclassified variables during the last 24 iterations. In this table, and in the following ones, k_f denotes the final iteration, and so $k_f - 1$ is the last but one iteration and so on.

We see that the Tapia indicator has a better behavior than the new indicator. Indeed, for the majority of test examples the Tapia indicator has less than 1% of misclassified variables in the last 5 iterations, whereas for the new indicator there is a considerable amount of problems

Table 6.1: Number of problems with less than 1% misclassified variables

It.	Tapia indicator	New indicator	Intersection
k_f	64	64	64
$k_f - 1$	64	61	64
$k_f - 2$	64	49	64
$k_f - 3$	62	34	62
$k_f - 4$	57	24	58
$k_f - 5$	40	20	52
$k_f - 6$	28	18	47
$k_f - 7$	17	15	40
$k_f - 8$	10	10	31
$k_f - 9$	5	9	28
$k_f - 10$	3	5	27
$k_f - 11$	1	4	22
$k_f - 12$	1	3	17
$k_f - 13$	1	3	13
$k_f - 14$	1	3	11
$k_f - 15$	1	2	8
$k_f - 16$	0	2	7
$k_f - 17$	0	1	5
$k_f - 18$	0	1	3
$k_f - 19$	0	0	3
$k_f - 20$	0	0	2
$k_f - 21$	0	0	2
$k_f - 22$	0	0	1
$k_f - 23$	0	0	1

with more than 1% misclassified variables even 3 or 4 steps before the final iteration.

The most interesting conclusion one can draw from Table 6.1, however, is the superior behaviour of the intersection indicator. In view of its very definition, it is clear that this indicator has less misclassified variables than the other two indicators, however, it is not clear, a priori, that such a good behavior could be obtained. Actually, the analysis of the complete numerical results [6] shows that the number of misclassified variables by the intersection indicator is very small at almost all iterations and for almost all test examples. To further illustrate the behavior of the intersection indicator, in Table 6.2, we report how many test problems have no misclassified variables in the last 10 iterations when using the intersection indicator.

The numbers provided by this table are still very encouraging and show that a suitable combination of the new and the Tapia indicator provides useful information.

Table 6.2: Number of problems with no misclassified variables (intersection indicator)

It.	k_f	k_f-1	k_f-2	k_f-3	k_f-4	k_f-5	k_f-6	k_f-7	k_f-8	k_f-9
# prob.	64	64	62	51	40	20	12	6	4	3

6.2 Correctly Identified Indices

As we already observed, this is the second index essential to assess the quality of an indicator. Table 6.3 shows for how many test examples we have at least 50% correctly identified indices in the last 15 iterations. We do not consider iterations before k_f-14 because, by the results reported in Table 6.1, before this iteration for most problems the number of misclassified indices is higher than 1% so that the information provided by the indicators is not reliable. Table 6.4 is analogous to Table 6.3 but in this case we consider problems for which all classified indices are correctly classified.

Table 6.3: Number of problems with 50% correct identification

It.	Tapia indicator	New indicator	Intersection
k_f	64	64	64
k_f-1	64	64	64
k_f-2	64	64	64
k_f-3	64	64	64
k_f-4	64	61	58
k_f-5	63	58	52
k_f-6	61	57	46
k_f-7	57	54	37
k_f-8	48	51	32
k_f-9	46	48	27
k_f-10	41	44	23
k_f-11	30	37	20
k_f-12	25	33	18
k_f-13	24	27	15
k_f-14	22	27	14

From Tables 6.3 and 6.4 we see that the new indicator and the Tapia one have a similar behavior, although the new indicator seems able to better classify indices in early stages while the Tapia indicator behaves better when close to a solution. By its very definition the intersection indicator is expected to have the worst behavior with respect to the percentage of correctly classified indices. However, the performance of this percentage is still more than acceptable and furthermore the results of this section should always be read in the light of the results of the previous section showing that the intersection indicator is "slower" than the other two, but more reliable.

Table 6.4: Number of problems with 100% correct identification

It.	Tapia indicator	New indicator	Intersection
k_f	64	64	64
$k_f - 1$	51	46	42
$k_f - 2$	18	14	13
$k_f - 3$	2	2	0

Looking at the complete results we may also note that there is a surprisingly high number of problems where more than 50% of indices are correctly classified already in the very first iterations.

6.3 Globally Correctly Identified Indices

Also for this quality index we report two tables to summarize the results. This quality index is similar to the previous one with a greater emphasis on stability of the indicators. To give the reader a different point of view, however, the tables we report have a different structure than those of Subsection 6.2. In Table 6.5 we report, for each indicator, the number of problems for which the percentage of globally correctly identified indices at the first iteration is between 0 and 10 percent, 10 and 20 percent and so on. The same kind of data is reported in Table 6.6 for the iteration $k_f - 4$.

Table 6.5: Globally correctly identified variables at first iteration

%	Tapia indicator	New indicator	Intersection
0–10	37	9	42
10–20	4	7	1
20–30	4	13	3
30–40	6	7	7
40–50	10	7	8
50–60	3	8	3
60–70	0	6	0
70–80	0	1	0
80–90	0	6	0
90–100	0	0	0

Obviously the *globally* correct classification of indices is more difficult than the simple correct identification of a certain number of indices. However the qualitative behavior that emerges from the two tables and also from the analysis of the complete numerical results is very similar to the one described in the previous subsection: The new indicator behaves (considerably) better in early stages than the Tapia indicator which, however, is superior

Table 6.6: Globally correctly identified variables at iteration $k_f - 4$

%	Tapia indicator	New indicator	Intersection
0–10	0	0	0
10–20	0	0	0
20–30	0	0	0
30–40	0	1	2
40–50	0	2	4
50–60	1	6	6
60–70	2	6	7
70–80	9	11	14
80–90	10	22	18
90–100	42	16	13

eventually. The intersection indicator is obviously worst than the other two indicators, even if not drastically so, but the information it provides should be regarded as more reliable.

7 Summary and Conclusions

In this paper we introduced a new technique for identifying the status of variables in the relative interior of the solution set of a column sufficient linear complementarity problem by using the information available in "nearby" points. The theoretical properties of the new indicator appear to be interesting. The technique we propose may be the only available option for some classes of problems (column sufficient LCP which are not monotone, for example) or algorithms (smoothing techniques, for example).

We tested the technique on linear programming problems, in an interior-point framework, and compared its behavior to the Tapia indicator. The results are encouraging and, in our opinion, indicate the practical viability of our approach. In particular, the combination of the new indicator with the Tapia one appears to be particularly promising. Since the computational cost of our technique is very low, this combination certainly deserves further study, at least in the linear programming case. However, the numerical results we reported should be regarded as preliminary. In fact, on the one hand the behavior of the new technique can probably be improved by considering different choices for the parameters α, θ (for example, a different α can be used for each index or in the definition of the sets $\mathcal{B}^k$, $\mathcal{N}^k$ and $\mathcal{J}^k$) and for the indicator function, on the other hand the behavior of the identification technique on wider classes of problems should also be investigated. To this end we may add that some very preliminary results with small, randomly generated LCPs seem promising.

Finally, the use of identification techniques to actually facilitate the solution of linear complementarity problems is an issue that certainly deserves careful examination and that we intend to study in the near future. We refer the interested reader to [3] for a good introduction to this kind of problems.

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