COMPLEMENTARITY AND RELATED PROBLEMS

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Abstract. This survey gives an introduction to some of the recent developments in the field of complementarity and related problems. After presenting two typical examples and the basic existence and uniqueness results, we focus on some new trends for solving nonlinear complementarity problems. Extensions to mixed complementarity problems, variational inequalities and mathematical programs with equilibrium constraints are also discussed.

Key Words. Nonlinear complementarity problem; Mixed complementarity problem; Variational inequalities; Mathematical programs with equilibrium constraints.

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1 Introduction

In this paper, we give a survey of complementarity problems that covers some standard results and presents a number of new developments in the field. These developments are first described for the most basic problem, the nonlinear complementarity problem (NCP), which is to find a vector $x \in \mathbb{R}^n$ satisfying the system of equations and inequalities

$$x \geq 0, \quad F(x) \geq 0, \quad x^T F(x) = 0$$

or, equivalently,

$$x_i \geq 0, \quad F_i(x) \geq 0, \quad x_i F_i(x) = 0 \quad \text{for } i = 1, \ldots, n.$$

Here $F : X \to \mathbb{R}^n$ is a given function defined on a subset $X \subseteq \mathbb{R}^n$ containing at least the nonnegative orthant.

Due to expository and space considerations, we do not always present a result in its most general form; for example, we will introduce the class of monotone functions in Section 3 and state a couple of important results for this class even though some of them actually hold for the somewhat broader classes of pseudomonotone or P_0-functions.

Many of the basic theoretical results in complementarity problems have been known for a long time; an excellent survey of much of the research in this area prior to 1990 can be found in [Harker and Pang, 1990]. Further references and more recent work can also be found in [Ferris and Pang, 1997a, Harker, 1993, Pang, 1994]. Here, we give just a flavor of the known existence and uniqueness results and concentrate more on the algorithmic point of view.

The organization of this paper is as follows. Section 2 gives two typical applications of complementarity problems as they arise in economics and engineering sciences. In Section 3 we then state a few existence and uniqueness results for the standard NCP, and Section 4 describes a couple of different solution techniques for NCP. Section 5 deals with some more general equilibrium-type problems, namely mixed complementarity problems, variational inequalities and mathematical programs with equilibrium constraints.

The notation used in this paper is assumed to be rather standard: Then n-dimensional Euclidean space is denoted by $\mathbb{R}^n$ with $\|x\|$ being the Euclidean norm of a vector $x \in \mathbb{R}^n$. The scalar product of two vectors $x, y \in \mathbb{R}^n$ is written as $x^T y$. An inequality $x \geq 0$ means that all components of the vector $x \in \mathbb{R}^n$ are nonnegative. If $F : \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable function, we denote its Jacobian at a point $x \in \mathbb{R}^n$ by $F'(x)$. Furthermore, if $M \in \mathbb{R}^{n \times n}$ denotes an arbitrary matrix with elements m_{ij} and $I, J \subseteq \{1, \ldots, n\}$ are two given subsets, then M_{IJ} is an $|I| \times |J|$-dimensional submatrix with entries m_{ij} for $i \in I$ and $j \in J$.

2 Examples

Complementarity problems arise frequently in the general equilibrium theory [Arrow and Debreu, 1954] from economics. Many complex computable general equilibrium models are used for various aspects of policy design and analysis, including carbon abatement [Manne and Rutherford, 1993] and trade reform [Harrison et al., 1997]. Other economic applications use game theory [Nash, 1950]; new examples of this are becoming popular due to deregularization of electricity markets [Smeers, 1997].

Our particular example is taken from [Scarf, 1973] and is a schematic Arrow-Debreu equilibrium problem [Arrow and Debreu, 1954]. The economy contains ℓ commodities, m utility-maximizing consumers and n profit-maximizing producers. The key variables of the model are the prices p of the commodities. Given these prices, it is assumed that the agents demand certain amounts of the commodities, which are supplied so as to maximize profit in each production sector. Complementarity is derived from Walras' law, namely that supply will equal demand in the equilibrium state.

The production side of the economy depends on a technology matrix

$$A = \left[\begin{array}{cccc} a^1 & a^2 & \cdots & a^n \end{array} \right] \in \mathbb{R}^{\ell \times n}$$

that determines the unit output level of each sector. Generally, the entries of A are nonlinear functions of the price vector p and can be calculated using production functions [see Ferris and Pang, 1997b]. For simplicity, we will assume here a constant technology matrix, and convert the activity level z_j of sector j into commodities using $a^j z_j$. Under the aforementioned assumption of price taking, each sector maximizes profits for given prices p:

$$\max\{p^T a^j z_j \;:\; z_j \geq 0\}.$$

The necessary and sufficient optimality conditions for production are therefore:

$$z \geq 0,\; A^T p \leq 0,\; p^T A z = 0. \tag{1}$$

For consumption, we derive a demand function for each of the consumers in the economy. The ith consumer is endowed with an initial wealth w_i and, assuming that he is also a price taker, will maximize his utility subject to his budget constraints. As is typical in such problems, the utility function is assumed to be either Cobb-Douglas

$$u^i(d) := \prod_c d_c^{\lambda_c}, \quad \lambda_c \geq 0, \quad \sum_c \lambda_c = 1,$$

or constant elasticity of substitution (CES)

$$u^i(d) := \left(\sum_{c=1}^{\ell} \lambda_c x_c^{\rho} \right)^{1/\rho}, \qquad \lambda_c \geq 0, \quad c = 1, \ldots, \ell,$$

where λ_c are given parameters dependent on the consumer i. The consumer solves

$$\max\{u^i(d^i) \;:\; p^T d^i \leq w_i,\; d^i \geq 0\}, \tag{2}$$

which in the case of Cobb–Douglas utility, gives

$$d_c^i(p, w_i) = \frac{\lambda_c w_i}{p_c}, \tag{3}$$

whereas for CES utility,

$$d_c^i(p, w_i) = \frac{(p_c/\lambda_c)^{r-1} w_i}{\sum_{j=1}^{\ell} p_j^r/\lambda_j^{r-1}}, \quad r = \rho/(\rho - 1). \tag{4}$$

Typically, wealth is determined from an initial endowment of commodities e^i, so that $w_i = p^T e^i$.

While the prices were assumed above to be given to each of the agents in the economy, the model actually uses Walras' law to determine the prices so that markets clear. This clearance condition is stated succinctly as follows:

$$p \geq 0,\ Az - \textstyle\sum_{i=1}^m (d^i(p, p^T e^i) - e^i) \geq 0,\ p^T(Az - \sum_{i=1}^m (d^i(p, p^T e^i) - e^i)) = 0.$$

If the price p is positive then supply $Az + \sum_{i=1}^m e^i$ must equal demand $\sum_{i=1}^m d^i(p, w_i)$. Otherwise, when excess supply is available, the price must be zero.

Summing all the information regarding production and consumption results in the following complementarity problem:

$$\begin{gathered} z \geq 0,\ -A^T p \geq 0,\ p^T Az = 0, \\ p \geq 0,\ Az - \textstyle\sum_{i=1}^m (d^i(p, p^T e^i) - e^i) \geq 0,\ p^T(Az - \sum_{i=1}^m (d^i(p, p^T e^i) - e^i)) = 0. \end{gathered}$$

Note that if (p, z) solves the above problem, then so does $(\lambda p, z)$ for any $\lambda > 0$. This frequently causes algorithms difficulty in determining a solution. Typically this singularity is removed by fixing a numeraire, that is fixing a price and dropping the corresponding complementary relationship.

The data for a particular instance of this model is available in MCPLIB [Dirkse and Ferris, 1995a]. Many extensions of this basic framework are possible [Ferris and Pang, 1997b, Rutherford, 1995, 1999]; examples using specialized software are described in [Ferris et al., 1999a, Ferris and Munson, 1999a, Rutherford, 1995, 1999].

Another example of the use of complementarity problems is in the field of structural analysis. One of the aims of structural analysis is to find displacement, stress and strain fields in a given structure under given conditions. Typically, loads F are specified and the problem is to find displacements u, stresses Q and strains q. Compatibility or kinematic relationships relate u and q, equilibrium or statics relationships relate F and Q and the constitutive laws relate Q and q. There are many models in use by engineers due to different assumptions for displacement and constitutive laws. For example, small or large displacements can be assumed, different discretization or finite element models can be applied and time independent plasticity relationships can be used or not.

Starting in the 1970's, Maier and colleagues [Maier, 1970, 1971, Maier and Novati, 1990, Tin-Loi and Misa, 1996, Tin-Loi and Pang, 1993] proposed the use of complementarity in mathematical models for structural mechanics. The general form of the model is as follows. The compatibility relationships are determined from the geometry of the structure using $q = q(u)$. The equilibrium relationships are normally given by

$$C^T Q = F,\ \ C = \frac{\partial q}{\partial u},$$

so that in a small displacement model $q = Cu$. The constitutive laws can be written

$$Q = Se, \quad p = N\lambda, \quad q = e + p,$$

where e and p represent elastic and plastic strains, S corresponds to elastic stiffnesses for each member of the structure, N contains unit outward normals and λ are the plastic multipliers. The yield function satisfies the following complementarity relationship

$$\lambda \geq 0, \quad -N^T Q + Q_* + h(\lambda) \geq 0, \quad \lambda^T(-N^T Q + Q_* + h(\lambda)) = 0.$$

The elastic limit Q_*, and h (representing softening behavior) are material properties and are typically given. Most of the previous work in this area has been concerned with linear models of this format; typically, many of the above variables are eliminated to construct a standard form linear complementarity model. A more reasonable modeling approach maintains the sparsity and the mixed complementarity nature of the above formulation (see Section 5.1 and [Tin-Loi and Ferris, 1997]). Nonlinear extensions of the models are detailed in [Bolzon et al., 1995, Ferris and Pang, 1997b, Tin-Loi and Pang, 1993].

3 Existence

In this section, we deal with the following questions: Does the NCP have at least one solution? When is the solution unique? What is the structure of the solution set?

In general, it is impossible to give a satisfactory answer to any of these questions without posing certain conditions on the function F. In order to give an idea of what can happen, consider the following simple examples.

Example 3.1 *In all examples given here, the function $F : \mathbb{R}^n \to \mathbb{R}^n$ is defined for $n = 1$.*

(a) If $F(x) = -x - 1$, then NCP has no solution since $F(x) < 0$ for all $x \geq 0$.

(b) If $F(x) = x - 1$, then NCP has a unique solution $x^* = 1$.

(c) If $F(x) \equiv 0$, then NCP has infinitely many solutions, namely the half line $[0, +\infty)$.

(d) If $F(x) = (x-1)^2 - 1$, then NCP has two isolated solutions, namely $x^* = 0$ and $x^{**} = 2$.

In order to state some existence and, possibly, uniqueness results, we now introduce some classes of functions which play the role of corresponding convexity properties in optimization problems.

Definition 3.2 *Let $X \subseteq \mathbb{R}^n$ be a nonempty set and $F : X \to \mathbb{R}^n$ be a given function. Then F is said to be*

(i) monotone *(on X) if $(x - y)^T(F(x) - F(y)) \geq 0$ for all $x, y \in X$;*

(ii) strictly monotone *(on X) if $(x - y)^T(F(x) - F(y)) > 0$ for all $x, y \in X$ with $x \neq y$;*

(iii) strongly monotone *(on X) if there exists a constant $\mu > 0$ (called* modulus*) such that* $(x-y)^T(F(x)-F(y)) \geq \mu \|x-y\|^2$ *for all $x, y \in X$.*

Using these definitions, we can state the following result.

Theorem 3.3 *Let $F : X \to \mathbb{R}^n$ be a continuous function on $X = \mathbb{R}^n_+$. Then the following statements hold:*

(a) If F is monotone on X, then NCP has a convex (possibly empty) solution set.

(b) If F is strictly monotone on X, then NCP has at most one solution.

(c) If F is strongly monotone on X, then NCP has a unique solution.

Unfortunately, the assumption in Theorem 3.3 (c) is rather strong and usually not satisfied in many practical applications. On the other hand, the monotonicity assumption in Theorem 3.3 (a) is a much weaker condition which, in fact, can be guaranteed to hold in quite a few situations. So it would be desirable to have an existence result also for monotone complementarity problems. Fortunately, such a result exists if, in addition, we assume that there is a strictly feasible point.

Definition 3.4 *A vector $\hat{x} \in \mathbb{R}^n$ is called* strictly feasible *for NCP if $\hat{x} > 0$ and $F(\hat{x}) > 0$.*

The strict feasibility assumption is closely related to the well-known Slater condition in convex optimization problems and allows us to state the following result.

Theorem 3.5 *Let $F : X \to \mathbb{R}^n$ be a continuous function on $X = \mathbb{R}^n_+$. Assume that F is monotone on X and there is a strictly feasible point $\hat{x}$ for NCP. Then NCP has a nonempty and compact solution set.*

Moré and Rheinboldt [1973] provide proofs of most of the results stated in this section, [see also Harker and Pang, 1990, Cottle et al., 1992, for some generalizations and specializations].

4 Algorithms

Here we want to describe the basic ideas of some solution methods for the nonlinear complementarity problem. Suitable algorithms should have the following desirable properties:

1. globally convergent
2. locally fast convergent.

The first requirement says that, if possible, we want to have a method which is guaranteed to find a solution of the NCP independent of how good or bad our starting point is. This can usually be guaranteed only if F has certain properties like being monotone or strongly monotone.

For the second point, a method is viewed as being locally fast convergent if it is locally superlinearly or quadratically convergent. Fast convergence can often be established under a condition called strong regularity [Robinson, 1980].

Definition 4.1 *Let x^* be a solution of NCP and define the index sets*

$$\alpha := \{i \mid x_i^* > 0 = F_i(x^*)\} \quad \text{and} \quad \beta := \{i \mid x_i^* = 0 = F_i(x^*)\}.$$

Then x^* is called a strongly regular *solution if the submatrix $F'(x^*)_{\alpha\alpha}$ is nonsingular and the Schur-complement*

$$F'(x^*)_{\alpha\cup\beta,\alpha\cup\beta}/F'(x^*)_{\alpha\alpha} := F'(x^*)_{\beta\beta} - F'(x^*)_{\beta\alpha}F'(x^*)_{\alpha\alpha}^{-1}F'(x^*)_{\alpha\beta} \in \mathbb{R}^{|\beta|\times|\beta|}$$

has positive principal minors.

The next subsections discuss several different algorithms. Most of these algorithms are based on a suitable reformulation of the complementarity problem as a nonsmooth equation.

4.1 Projection-Type Methods

Projection-type methods are based on the following simple observation.

Proposition 4.2 *A vector x^* is a solution of NCP if and only if x^* satisfies the fixed-point equation*

$$x = (x - \gamma F(x))_+,$$

where $\gamma > 0$ is an arbitrary constant and z_+ denotes the Euclidean projection of a vector $z \in \mathbb{R}^n$ onto the nonnegative orthant.

Projection-type methods typically attempt a fixed-point iteration. The simplest consists of the iteration

$$x^{k+1} := (x^k - \gamma F(x^k))_+, \quad k = 0, 1, 2, \dots, \tag{5}$$

where $x^0 \in \mathbb{R}^n_+$ is any given starting point [see Auslender, 1976, Bertsekas and Tsitsiklis, 1989, Sibony, 1970]. Using Banach's fixed point theorem, it follows easily that this method is globally convergent if F is strongly monotone and globally Lipschitzian provided the constant $\gamma > 0$ is sufficiently small. Furthermore, the rate of convergence one can expect is linear.

Modifications of this projection method have better global convergence properties. For example, the extragradient method [Khobotov, 1987, Korpelevich, 1976, Marcotte, 1991, Nagurney, 1993] generates iterates using the formula

$$x^{k+1} := (x^k - \gamma F((x^k - \gamma F(x^k))_+))_+, \quad k = 0, 1, 2, \dots, \tag{6}$$

i.e., the difference with (5) is that the function F is not evaluated at the previous iterate x^k, but at a point which would already be the next iterate for (5).

The extragradient method can be shown to be globally convergent if F is monotone (rather than strongly monotone) and globally Lipschitzian provided the constant $\gamma > 0$ is sufficiently small [see Nagurney, 1993].

In fact, it is even possible to remove the assumptions that F is globally Lipschitzian and $\gamma > 0$ has to be sufficiently small (in practice, it is usually not known how small γ has to be). This is usually done by choosing γ dynamically in a suitable line search within the extragradient method [see Solodov and Tseng, 1996, Sun, 1996, Iusem and Svaiter, 1997, Solodov and Svaiter, 1999].

These modified extragradient methods have nice global convergence properties for monotone complementarity problems, but they are at best linearly convergent. Due to their slow convergence, we recommend their use only if either the complementarity problem is huge or if it is impractical to form F'.

4.2 Merit Functions

We first give the definition of a merit function as it will be used within this survey.

Definition 4.3 *A function* $\Psi : \mathbb{R}^n \to \mathbb{R}$ *is called a* merit function *for NCP if it has the following properties:*

(a) $\Psi(x) \geq 0$ *for all* $x \in \mathbb{R}^n$*;*

(b) $\Psi(x) = 0$ *if and only if* x *is a solution of NCP.*

This definition suggests replacing the NCP by the unconstrained minimization problem

$$\min \Psi(x), \quad x \in \mathbb{R}^n.$$

In order to apply standard optimization tools to this minimization problem, the merit function should satisfy certain smoothness conditions. In fact, the merit function Ψ should be at least once continuously differentiable, if possible even twice. It turns out, however, that most known merit functions are only once continuously differentiable. For example, the implicit Lagrangian [Mangasarian and Solodov, 1993]

$$\Psi(x) := \sum_{i=1}^{n} \left\{ x_i F_i(x) + \frac{1}{2\alpha} \left(\max{}^2\{0, x_i - \alpha F_i(x)\} - x_i^2 + \max{}^2\{0, F_i(x) - \alpha x_i\} - F_i(x)^2 \right) \right\} \tag{7}$$

(where $\alpha > 1$ is a fixed parameter), the Fischer-Burmeister merit function [Facchinei and Soares, 1997, Geiger and Kanzow, 1996]

$$\Psi(x) := \frac{1}{2} \sum_{i=1}^{n} \left(\sqrt{x_i^2 + F_i(x)^2} - x_i - F_i(x) \right)^2 \tag{8}$$

or the penalized Fischer-Burmeister merit function [Chen et al., 1997]

$$\Psi(x) := \frac{1}{2} \sum_{i=1}^{n} \left\{ \lambda \left(\sqrt{x_i^2 + F_i(x)^2} - x_i - F_i(x) \right) - (1 - \lambda) \max\{0, x_i\} \max\{0, F_i(x)\} \right\}^2 \tag{9}$$

(where $\lambda \in (0, 1)$ is a fixed parameter) are all known to be merit functions that are once but not twice continuously differentiable.

This is one of the reasons why we do not suggest minimizing a merit function directly using existing unconstrained optimization software. Another reason is that most merit functions can be derived from corresponding equation reformulations (see the following subsection), and it is the interplay between such an equation reformulation and its merit function that makes these methods successful.

Apart from this, merit functions are often used in order to monitor global convergence of different algorithms. We believe this is actually the major contribution of a merit function.

We close this subsection by stating two types of results for the above mentioned three merit functions. The first result indicates sufficient conditions to ensure that a stationary point of the merit function Ψ is a global minimum and hence a solution of NCP.

Theorem 4.4 *Assume that $F : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable. If F is strongly monotone, then every stationary point of the implicit Lagrangian (7) is a solution of NCP. The same statement holds for the Fischer-Burmeister merit function (8) and the penalized Fischer-Burmeister merit function (9) under the weaker assumption that F is monotone.*

Most algorithms guarantee that any accumulation point of the iterates is at least a stationary point of Ψ and therefore, under the conditions of Theorem 4.4, a solution of the underlying complementarity problem. The next result concerns the compactness of the level sets

$$\mathcal{L}(c) := \{x \in \mathbb{R}^n \mid \Psi(x) \leq c\}$$

for a given constant c. This is a critical result, since a compact level set implies that the sequence of iterates will have an accumulation point.

Theorem 4.5 *Assume that $F : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and let $c \in \mathbb{R}$ be any given constant. If F is strongly monotone, then the level sets $\mathcal{L}(c)$ for any of the three merit functions (7)–(9) are compact (possibly empty). The same statement holds for the penalized Fischer-Burmeister merit function (9) under the weaker assumption that F is monotone and NCP has a strictly feasible point.*

There have been many other merit functions proposed in the literature, see e.g. [Luo and Tseng, 1997, Kanzow et al., 1997, Kanzow and Kleinmichel, 1998, Sun and Qi, 1998]. Currently, the penalized Fischer-Burmeister merit function and those in [Kanzow et al., 1997, Sun and Qi, 1998] have the strongest properties.

4.3 Equation-Based Methods

It is possible to reformulate the NCP as a nonlinear system of equations using so-called NCP-functions.

Definition 4.6 *A function $\phi : \mathbb{R}^2 \to \mathbb{R}$ is called an* NCP-function *if it satisfies the condition*

$$\phi(a, b) = 0 \iff a \geq 0, b \geq 0, ab = 0.$$

The following functions are NCP-functions:

(a) $\phi(a,b) = \min\{a,b\}$;

(b) $\phi(a,b) = \sqrt{a^2+b^2} - a - b$;

(c) $\phi(a,b) = \lambda\left(\sqrt{a^2+b^2} - a - b\right) - (1-\lambda)\max\{0,a\}\max\{0,b\}$ with $\lambda \in (0,1)$ fixed;

(d) $\phi(a,b) = -ab + \frac{1}{2}\min^2\{0, a+b\}$.

Normally (a) is called the minimum function, and (b) is called the Fischer-Burmeister function (it first appeared in [Fischer, 1992] where it is attributed to Burmeister). Example (c) is termed the penalized Fischer-Burmeister function [Chen et al., 1997]. Example (d) is the only NCP-function listed that is differentiable on the whole space $\mathbb{R}^2$ [Evtushenko and Purtov, 1984]. Many other NCP-functions are known in the literature [see Mangasarian, 1976, Sun and Qi, 1998].

For any given NCP-function ϕ, define $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\Phi(x) := \begin{pmatrix} \phi(x_1, F_1(x)) \\ \vdots \\ \phi(x_n, F_n(x)) \end{pmatrix}. \tag{10}$$

We call Φ the equation operator corresponding to ϕ. As an immediate consequence of the definition of Φ, we obtain the following characterization of the complementarity problem.

Proposition 4.7 *Let ϕ be an NCP-function and Φ be the corresponding equation operator defined by (10). Then x^* is a solution of NCP if and only if x^* is a solution of the nonlinear system of equations $\Phi(x) = 0$.*

If F and the NCP-function ϕ are continuously differentiable, then the same holds for the equation operator Φ. In this case, we can apply the classical Newton method to $\Phi(x) = 0$ to solve the underlying NCP. This leads to the iteration

$$x^{k+1} := x^k - \Phi'(x^k)^{-1}\Phi(x^k), \quad k = 0, 1, 2, \ldots$$

for a given starting point $x^0 \in \mathbb{R}^n$. We can expect local fast convergence if the standard assumptions for the Newton method are satisfied; in particular, the Jacobian matrix $\Phi'(x^*)$ has to be nonsingular at a solution x^* of NCP. Unfortunately, the following result from [Kanzow and Kleinmichel, 1995] shows that this matrix is singular at any degenerate solution. (Here, a solution x^* is called *degenerate* if the index set β from Definition 4.1 is nonempty.)

Proposition 4.8 *Let x^* be a degenerate solution of NCP. Assume that F and the NCP-function ϕ are differentiable, and let Φ denote the corresponding equation operator. Then the Jacobian $\Phi'(x^*)$ is singular.*

One way to overcome the singularity problem is to use a nondifferentiable NCP-function. This, in general, leads to a nondifferentiable equation operator Φ. However, Φ is usually at least locally Lipschitzian, so Φ is almost everywhere differentiable by a result due to

Rademacher. Let us denote by D_Φ the set of differentiable points of Φ. Then we define the so-called *B-subdifferential* [Robinson, 1987]

$$\partial_B \Phi(x) := \{H \in \mathbb{R}^{n\times n} \mid \exists \{x^k\} \subseteq D_\Phi : x^k \to x \text{ and } \Phi'(x^k) \to H\}$$

as well as its convex hull

$$\partial \Phi(x) := \operatorname{conv}\{\partial_B \Phi(x)\}$$

which is called the *generalized Jacobian* [Clarke, 1983]. Elements from $\partial_B \Phi(x)$ or $\partial \Phi(x)$ can be used instead of the classical (and in our situation sometimes not existing) Jacobian $\Phi'(x)$, and a typical nonsmooth Newton method then leads to the iteration

$$x^{k+1} := x^k - H_k^{-1}\Phi(x^k), \quad k = 0, 1, 2, \dots, \tag{11}$$

where H_k is an arbitrary element from $\partial_B \Phi(x^k)$.

Although, in general, it is difficult to compute elements from the B-subdifferential, we are interested only in particular operators Φ, and for those it is usually not difficult to compute an element $H_k \in \partial_B \Phi(x)$ [see, e.g. De Luca et al., 1996, Chen et al., 1997].

If Φ is the equation operator corresponding to either the minimum function, the Fischer-Burmeister function or the penalized Fischer-Burmeister function and x^* is a strongly regular solution of NCP, then it is possible to show that

(a) all elements $H \in \partial_B \Phi(x^*)$ are nonsingular, and

(b) Φ satisfies further smoothness assumptions (called *semismoothness/strong semismoothness* in [Qi and Sun, 1993]).

These two requirements are exactly those which are needed in order to prove local superlinear or quadratic convergence of the iterates generated by (11) [see Qi and Sun, 1993, Qi, 1993].

Furthermore, it is often relatively easy to make these methods globally convergent at least for monotone problems. To this end, one typically uses the vector

$$d^k := -H_k^{-1}\Phi(x^k)$$

as a search direction and performs a line search along this direction to get sufficient decrease in the corresponding merit function

$$\Psi(x) := \frac{1}{2}\Phi(x)^T\Phi(x). \tag{12}$$

Such a decrease is possible if the merit function is continuously differentiable [De Luca et al., 1996]. This condition is satisfied for the Fischer-Burmeister function and the penalized Fischer-Burmeister function discussed in the previous subsection. In general, these methods work well even for nonmonotone complementarity problems.

Unfortunately, the merit function Ψ corresponding to the minimum function is not everywhere differentiable, and therefore the nonsmooth Newton iteration (11) based on the minimum function is more difficult to globalize. The references [Pang, 1990, 1991, Pang and Gabriel, 1993, Gabriel and Pang, 1992, De Luca et al., 1999] give possible strategies.

4.4 Smoothing and Interior-Point Methods

Given an NCP-function ϕ and the corresponding equation operator Φ defined by (10), Proposition 4.8 suggests using nondifferentiable NCP-functions and nonsmooth algorithms. Another possibility to deal with nondifferentiable functions is the smoothing technique whose basis will be described in this section.

Smoothing methods for the NCP typically approximate a nonsmooth NCP-function ϕ by a smooth function ϕ_μ that depends on a smoothing parameter $\mu > 0$. For example, it is possible to approximate the Fischer-Burmeister function by

$$\phi(a,b) = \sqrt{a^2+b^2} - a - b \approx \sqrt{a^2+b^2+\mu} - a - b =: \phi_\mu(a,b). \tag{13}$$

Using similar techniques, it is possible to smooth most known NCP-functions.

As soon as we have such a smooth approximation ϕ_μ, we define the corresponding equation operator

$$\Phi_\mu(x) := \begin{pmatrix} \phi_\mu(x_1, F_1(x)) \\ \vdots \\ \phi_\mu(x_n, F_n(x)) \end{pmatrix}$$

which is now a smooth approximation of Φ if F itself is smooth.

The main idea of any smoothing method is to apply a standard Newton method to the nonlinear system of equations $\Phi_\mu(x) = 0$ and to drive the smoothing parameter μ down to zero. If we denote the value of μ at the kth iteration by μ_k, a typical iteration of a smoothing method is given by

$$x^{k+1} = x^k - \Phi'_{\mu_k}(x^k)^{-1}\Phi_{\mu_k}(x^k), \quad k = 0,1,2,\ldots$$

A critical point of all smoothing methods is the way the smoothing parameter μ_k gets updated at each iteration. If this is done in an appropriate way, it is possible to prove global convergence for monotone problems, and even local fast convergence if a solution x^* of the NCP satisfies the strong regularity assumption, [see Burke and Xu, 1998, Chen and Chen, 1999, Chen et al., 1998, Chen and Ye, 1999, Xu and Burke, 1998, for a couple of different smoothing methods].

Smoothing methods have an interesting connection to interior-point methods as pointed out in [Kanzow, 1996]. For example, let ϕ_μ denote the function defined in (13). Then it is easy to see that

$$\phi_\mu(a,b) = 0 \iff a > 0, b > 0, ab = \mu.$$

Hence solving the nonlinear system of equations

$$\Phi_\mu(x) = 0 \tag{14}$$

is equivalent to solving the central path conditions

$$x_i > 0, \quad F_i(x) > 0, \quad x_i F_i(x) = \mu \quad \forall i = 1, \ldots, n \tag{15}$$

which are typically used as the basis of path-following interior-point methods. However, the way smoothing methods and interior-point methods solve their (theoretically equivalent) subproblems (14) and (15), respectively, is completely different. Furthermore, there exist smooth approximations ϕ_μ of NCP-functions ϕ that are not related to the central path conditions [see Chen and Mangasarian, 1996].

So far, the theory developed for interior-point methods [see, e.g., the books Wright, 1997, Ye, 1997, Roos et al., 1997] is much stronger than the one developed for smoothing methods. For example, interior-point methods can often be shown to have polynomial complexity and local fast convergence even in the absense of the strong regularity condition. On the other hand, smoothing-type methods seem to be quite promising from a numerical point of view [see Chen and Mangasarian, 1996, Billups et al., 1997, Kanzow and Pieper, 1999]. We therefore expect further theoretical improvements in the field of smoothing methods in the near future.

4.5 Linearization Methods

Here, the fundamental idea is to linearize F about the current iterate x^k and solve the corresponding linear complementarity problem

$$x \geq 0, \quad F(x^k) + F'(x^k)(x - x^k) \geq 0, \quad x^T \left(F(x^k) + F'(x^k)(x - x^k)\right) = 0 \tag{16}$$

to generate x^{k+1}. In the context of generalized equations, this method, which has been commonly termed SLCP (for sequential LCP), was shown to be locally well-defined and fast convergent under the strong regularity condition [Josephy, 1979]. Mathiesen [1987, 1985] successfully used this scheme for solving equilibrium models such as those outlined in Section 2. While this method works extremely well in the neighborhood of a solution, there are substantial difficulties in globalizing the approach. Two key difficulties are subproblems not having solutions and choosing appropriate merit functions that do not destroy fast local convergence. It is no mere coincidence that these problems plague sequential quadratic programming methods too – their theory can be treated analogously.

For strongly monotone variational inequalities, convergence of line search schemes based on (16) are shown in [Fukushima, 1992, Taji et al., 1993]. A theoretically more general approach is via reformulation of the problem as a nonsmooth system of equations. The normal map [Robinson, 1992] defined as

$$F_+(y) := F(y_+) + y - y_+ \tag{17}$$

is equivalent to solving the NCP in the following sense. If y is a zero of (17) then y_+ solves the NCP. Furthermore, if x is a solution of the NCP, then $x - F(x)$ is a zero of the normal map. Note two points that contrast the normal map from previously mentioned nonsmooth equivalents of the NCP. First, it is defined for all points in $\mathbb{R}^n$ even if F is only defined on $\mathbb{R}^n_+$. It is known to be a piecewise smooth map, the "pieces" correspond to the orthants of $\mathbb{R}^n$, a subdivided manifold [Eaves, 1976], typically called the normal manifold. Secondly, the nonsmoothness is inside the function F as opposed to around the function F, so that F is only evaluated on the positive orthant.

The Newton method proposed in [Robinson, 1994], in the context of the normal map, approximates $F_+(y)$ with a piecewise linear normal map

$$A^k(y) := F(y^k_+) + F'(y^k_+)(y_+ - y^k_+) + y - y_+.$$

The Newton point is a zero of A^k. It is well known that this is equivalent to SLCP by taking $x^k = y^k_+$. However, Ralph [1994] proposed a globalization scheme that extends damped Newton methods for smooth equations to the above situation. Instead of using Lemke's method [Cottle and Dantzig, 1968, Lemke and Howson, 1964] to solve (16), a path $p^k(t)$ is constructed [Dirkse and Ferris, 1995b] from y^k to y^{k+1} with the property that

$$A^k(p^k(t)) = (1-t)F_+(y^k).$$

This path has exactly the same properties that are required for standard line search methods, namely that small steps guarantee sufficient decrease. That theory can be extended to this case, provided that $\|F_+(y)\|$ is used as the merit function, to prove standard global and local convergence results [Dirkse and Ferris, 1995b, Ralph, 1994].

Considerable success has been reported for an implementation of this approach called the PATH solver [Dirkse and Ferris, 1995b, Ferris and Munson, 1999b]. This implementation uses standard large scale linear programming (simplex) technology [Nazareth, 1987] to generate the required path. SLCP methods typically use similar technology, except the subproblems use ray starts in Lemke's method as opposed to the required regular point starts in PATH. Some advanced techniques for choosing the initial piece of the normal manifold can be found in [Anstreicher et al., 1992, Dirkse and Ferris, 1997].

Most of the above implementations rely on heuristics to deal with the case where the subproblems have no solution. Recent work in this area has extended the global convergence theory along different lines. Ferris et al. [1999b] proposed a theoretical framework for globalization of feasible descent methods that uses (8) as a merit function. The main features of this framework are: (a) it is well-defined for an arbitrary complementarity problem, (b) it generates only feasible iterates, (c) it has a strong global convergence theory, and (d) it is locally fast convergent under standard regularity assumptions. The key observations of such a globalization are that if the subproblems are unsolvable, then a projected gradient step can be taken for the merit function, guaranteeing progress and that eventually the full Newton step will be accepted. Numerical results in the cited paper indicate this framework leads to substantial computational improvements.

5 Extensions

5.1 Mixed Complementarity Problems

While most of this paper has been written in the context of nonlinear complementarity problems, many practical applications generate problems for which some of the variables are nonnegative, others are bounded, and others are completely free. To accomodate these problems, many researchers have investigated the mixed complementarity problem (MCP), of finding a vector $x \in [l, u]$ such that exactly one of the following holds:

$$\begin{aligned} x_i = l_i \quad &\text{and} \quad F_i(x) > 0, \\ x_i = u_i \quad &\text{and} \quad F_i(x) < 0, \\ x_i \in [l_i, u_i] \quad &\text{and} \quad F_i(x) = 0. \end{aligned}$$

Note that l_i and u_i can be infinite. Two special cases are of importance. When $l_i = -\infty$ and $u_i = \infty$ then the MCP is exactly the problem of finding a zero of F. This shows why many of the techniques for solving complementarity problems are closely related to nonlinear equation theory, as opposed to standard optimization techniques. Furthermore, when $l_i = 0$ and $u_i = \infty$ the NCP is recovered, and thus it is clear that MCP is a generalization of NCP. Although there are many computational issues related to exploiting this framework, it is well known that appropriate changes of variables can reduce the MCP to a (larger dimensioned) NCP. Therefore, for theoretical purposes it is entirely adequate to deal exclusively with NCP's. In practice, however, exploiting the MCP structure is of paramount importance [Dirkse and Ferris, 1995a, Ferris and Pang, 1997b].

5.2 Variational Inequalities

Given a nonempty, closed and convex subset $X \subseteq \mathbb{R}^n$ and a function $F : X \to \mathbb{R}^n$, the variational inequality problem (VIP) is to find a vector $x^* \in \mathbb{R}^n$ such that

$$F(x^*)^T(x - x^*) \geq 0 \quad \text{for all } x \in X.$$

It is not difficult to see that the variational inequality problem VIP reduces to NCP if the feasible set X is equal to the nonnegative orthant $\mathbb{R}^n_+$, and becomes the MCP if $X = [l, u]$.

Many results and algorithms described above for the complementarity problem can actually be extended to the more general variational inequality problem. For example, Theorem 3.3 still holds if NCP is replaced everywhere by VIP. Also Theorem 3.5 has an analogous result for variational inequalities [Harker and Pang, 1990].

For solution techniques, all the projection–type methods outlined in Section 4.1 can be applied to VIP and have the same global convergence properties if we interpret the symbol z_+ as the Euclidean projection of a vector $z \in \mathbb{R}^n$ onto the feasible set X (rather than the projection onto $\mathbb{R}^n_+$).

Also the linearization methods have a direct translation for variational inequalities. For example, the SLCP method for VIP generates a sequence $\{x^k\}$ in such a way that x^{k+1} is a solution of the linearized variational inequality

$$\left(F(x^k) + F'(x^k)(x - x^k)\right)^T (y - x) \geq 0 \quad \text{for all } y \in X,$$

with similar convergence to that described in Section 4.5.

Furthermore, there exist a couple of gap function approaches for the solution of VIP which may be viewed as a generalization of the merit function approaches used for complementarity problems although the situation is much more complicated for VIP.

[Auslender, 1976] introduced the *gap function*, defined as

$$g(x) := \sup_{y \in X} \left\{ F(x)^T(x - y) \right\}.$$

It is not difficult to see that g is nonnegative on the feasible set X and that $g(x) = 0$ and $x \in X$ if and only if x solves VIP. Hence VIP can be rewritten as the constrained minimization problem

$$\min g(x) \quad \text{s.t.} \quad x \in X.$$

Some algorithms based on this are given in [Marcotte, 1985, Marcotte and Dussault, 1987]. However, the gap function g is not differentiable in general. Moreover, when X is unbounded, g is in general not finite-valued.

These drawbacks can be avoided by using the *regularized gap function* [Auchmuty, 1989, Fukushima, 1992]

$$f_\alpha(x) := \max_{y \in X} \left\{ F(x)^T(x-y) - \frac{\alpha}{2} \|x-y\|^2 \right\},$$

where α is any positive parameter. It is also nonnegative on the set X and x solves VIP if and only if $f_\alpha(x) = 0$ and $x \in X$. Hence, VIP is equivalent to the constrained optimization problem

$$\min f_\alpha(x) \quad \text{s.t.} \quad x \in X. \tag{18}$$

In addition, f_α can be shown to be continuously differentiable and finite-valued everywhere.

However, the regularized gap function has a serious drawback: If x^* denotes a constrained stationary point of (18), then F has to be strongly monotone in order to guarantee that x^* is a solution of VIP. Another extension, the *D-gap function* [Peng, 1997, Yamashita et al., 1997] also suffers from the same drawback.

5.3 Mathematical Programs with Equilibrium Constraints

A mathematical program with equilibrium constraints (MPEC) is a constrained optimization problem with some of the constraints representing certain equilibrium conditions. For example, if these equilibrium conditions are described by a complementarity problem, then an MPEC can be formulated as

$$\min f(x,y) \quad \text{s.t.} \quad x \in X,\ y \geq 0,\ F(x,y) \geq 0,\ y^T F(x,y) = 0; \tag{19}$$

here, $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ are given functions and $X \subseteq \mathbb{R}^n$ is an abstract constraint which can usually be represented by standard equality/inequality constraints.

Note that the MPEC (19) has two types of variables, the design variables x and the state variables y. The complementarity constraints in (19) are parameterized in the design variables. While extensions or particular instantiations of (19) are possible, we will restrict ourselves to this form in order to convey the main algorithmic ideas.

The first point is that (19) should not be treated as a standard constrained optimization problem, since it is well known that it does not even satisfy the Mangasarian-Fromovitz constraint qualification [Chen and Florian, 1995]. Therefore, standard methods from constrained optimization are prone to failure.

Several researchers have proposed more specialized solution methods for MPEC problems. Given there is much current research in this area, we will only summarize a few central ideas.

One way to overcome the previously mentioned difficulty in (19) is to add the constraint $y^T F(x,y) = 0$ to the objective function and then to solve a sequence of penalized problems

$$\min f(x,y) + \rho y^T F(x,y) \quad \text{s.t.} \quad x \in X,\ y \geq 0,\ F(x,y) \geq 0$$

with penalty parameter $\rho > 0$. In general, this gives only an inexact penalty reformulation unless some strong conditions hold [Luo et al., 1996, Mangasarian and Pang, 1997, Scheel and Scholtes, 1998, Scholtes and Stöhr, 1999]. Some researchers also proposed exact penalty functions for the MPEC (19) which, however, are severely nondifferentiable and therefore difficult to deal with from a numerical point of view.

Another technique for the solution of the MPEC (19) is the implicit programming approach. These methods assume that, for any given $x \in X$, one can find a solution $y(x)$ of the complementarity constraints in (19). They replace the variable y by the function $y(x)$ in the objective function f and try to solve the optimization problem

$$\min f(x, y(x)) \quad \text{s.t.} \quad x \in X \tag{20}$$

which is a problem in the design variables only. The function $y(x)$ is usually nondifferentiable, so one has to use nondifferentiable optimization methods to solve (20) [see Outrata, 1994, Outrata and Zowe, 1995].

Another possibility to deal with the MPEC (19) is to reformulate the complementarity constraints using an NCP-function ϕ. One obtains a constrained optimization problem of the form

$$\min f(x, y) \quad \text{s.t.} \quad x \in X,\ \phi(y_i, F_i(x, y)) = 0 \quad (i = 1, \dots, m).$$

In order to avoid the nonsmooth constraints $\phi(y_i, F_i(x, y)) = 0$, we can use the smoothing idea and approximate this problem by the smooth optimization problem

$$\min f(x, y) \quad \text{s.t.} \quad x \in X,\ \phi_\mu(y_i, F_i(x, y)) = 0 \quad (i = 1, \dots, m),$$

where, of course, ϕ_μ is a smooth approximation for ϕ, see e.g., (13). This idea has been exploited in [Facchinei et al., 1999, Fukushima et al., 1998, Jiang and Ralph, 1997]. Alternatively, and closely related to the smoothing idea, we can apply interior-point techniques to the perturbed problem

$$\min f(x, y) \quad \text{s.t.} \quad x \in X,\ y_i > 0, F_i(x, y) > 0,\ y_i F_i(x, y) = \mu \quad (i = 1, \dots, m).$$

For some further techniques and several theoretical results about MPECs, we refer the reader to [Luo et al., 1996, Outrata et al., 1998].

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