COMPLEMENTARITY PROBLEMS IN ENGINEERING MECHANICS: MODELS AND SOLUTION

F. Tin-Loi[1] and M. C. Ferris[2]

[1] School of Civil and Environmental Engineering,
University of New South Wales,
Sydney 2052, Australia
[2] Computer Sciences Department,
University of Wisconsin – Madison,
Wisconsin 53706, USA

ABSTRACT

A large class of problems in engineering mechanics involves a so-called "complementarity" relationship representing the orthogonality of two sign-constrained vectors. Typical instances are plasticity laws and contact-like conditions. For state problems, the formulation leads to a mixed complementarity problem (MCP) whereas in synthesis (e.g. minimum weight design) or identification problems, a mathematical program with equilibrium constraints (MPEC) is obtained. The aim of this paper is two-fold. Firstly, it describes, through two typical models, how some important engineering mechanics problems can be formulated elegantly and naturally as either an MCP or an MPEC. Secondly, it describes a powerful computer-oriented environment for constructing and solving these mathematical programming problems, with features such as sparsity and automatic differentiation facilities being transparently accessible. This involves the use of the modeling language GAMS (an acronym for General Algebraic Modeling System) and its associated mathematical programming solvers (e.g. the industry standard MCP solver PATH). A simple generic model suitable for solving the state problem for trusses is used to clarify the syntax of GAMS models and to illustrate the ease with which they can be built and solved.

KEYWORDS

Complementarity, computational mechanics, optimization, plasticity, mathematical programming, modeling system.

INTRODUCTION

Complementarity, namely the requirement that two sign-constrained vectors are orthogonal, is a typical and recurrent mathematical structure of many state, design and inverse problems in nonlinear mechanics. This was first recognized in the late 1960s by Maier whose seminal work led some years later to the NATO conference "Engineering Plasticity by Mathematical Programming" (Cohn and Maier 1979), with important contributions by prominent researchers from both engineering and mathematical programming communities. The proceedings of that workshop still represent a valuable source of information on the elegant and powerful framework provided by mathematical programming, in particular complementarity, to discrete plasticity. More recent sources of reference, specifically on some engineering and economic applications of complementarity problems, are the review paper by Ferris and Pang (1997a) and the proceedings of the first "International Conference on Complementarity Problems" (Ferris and Pang 1997b).

To engineers, the study and application of complementarity notions in mechanics should have a twofold appeal: a refined mathematical formalism rich in useful theoretical results and a wealth of efficient and robust numerical algorithms. Unfortunately, the application of such concepts has been sporadic and below expectation. For instance, plasticity problems (involving a set of complementarity conditions between yield functions and plastic strains) are still largely solved through the iterative use of linear solvers when a complementarity formulation would automatically choose which inequalities to satisfy as equations.

The motivation of this paper is to show how modeling systems and their associated mathematical programming solvers can help with model building and solution of a number of important engineering mechanics problems, all characterized by the presence of complementarity conditions. In particular, we adopt the well-known governing equations of a simple holonomic (path-independent or nonlinear elastic) elastoplastic discrete model (a) to show how a state problem can be formulated as a mixed complementarity problem (MCP); (b) to formulate a minimum weight problem as an instance of a mathematical program with equilibrium (complementarity in our case) constraints (MPEC); and (c) to describe how the modeling system GAMS, an acronym for General Algebraic Modeling System (Brooke *et al.* 1992), can be used to model and solve, using the industry standard MCP solver PATH (Dirkse and Ferris 1995a), a simple example of the state problem for elastoplastic trusses.

GOVERNING RELATIONS FOR DISCRETE HOLONOMIC PLASTICITY

We refer to a suitably space-discretized structural system, the constituents (finite elements) of which obey holonomic plasticity laws. The governing relations for the whole structure can be elegantly expressed through generalized variables (see e.g. Cohn and Maier 1979) as follows:

$$f = C^T x, \quad q = Cv, \tag{1,2}$$

$$q = e + p, \quad x = Se, \quad p = Nz, \tag{3,4,5}$$

$$w \equiv -N^T x + Hz + r \geq 0, \quad z \geq 0, \quad w^T z = 0. \tag{6}$$

Vector and matrix quantities represent the unassembled contributions of corresponding elemental entities as concatenated vectors and block-diagonal matrices, respectively. For a structure with *d* degrees of freedom and *m* member generalized quantities, Eqn. 1 expresses equilibrium, through compatibility matrix $C \in \Re^{m \times d}$, between the nodal loads $f \in \Re^{d}$ and the natural stresses $x \in \Re^{m}$. Eqn. 2 describes linear compatibility of strains $q \in \Re^{m}$ with the nodal displacements $v \in \Re^{d}$. Relations 3-6 embody the holonomic constitutive laws. The additivity of elastic $e \in \Re^{m}$ and plastic $p \in \Re^{m}$ strains is given by Eqn. 3. Linear elasticity is represented in Eqn. 4, where $S \in \Re^{m \times m}$ collects unassembled element stiffnesses. Plastic strains *p* are defined in Eqn. 5 by an associated flow rule in term of the plastic multipliers $z \in \Re^{y}$ (*y* = number of yield functions) through the matrix of outward normals $N \in \Re^{m \times y}$ to the yield surface. Finally, we define in Eqn. 6 a linear (yield) function $w(x(z), z) : \Re^{y} \to \Re^{y}$ which is complementary with the nonnegative plastic multiplier vector z and which accommodates, through $H \in \Re^{y \times y}$, a class of hardening models with yield limits $r \in \Re^{y}$.

COMPLEMENTARITY MODELS

Based on the governing relations given by Eqns. 1-6, we are now in a position to formulate two types of mathematical problems involving complementarity. As representative models, we describe a standard state problem cast as an MCP and then a minimum weight synthesis problem as an MPEC.

State Problem (MCP)

The holonomic state problem requires the calculation of the state variables (x, v, z) for a given structure (i.e. for specified material and geometric properties) and loading. Since we intend to use the GAMS modeling system for solving this problem, we adopt a "mixed" formulation involving both static (x) and kinematic variables (v, z) (at variance with the usual approach of using z variables only).

After some obvious substitutions, the problem then becomes one of finding (x, v, z) from the following relations:

$$\begin{gathered} C^{T} x - f = 0, \\ S^{-1} x - C v + N z = 0, \\ w \equiv -N^{T} x + H z + r \geq 0, \quad z \geq 0, \quad w^{T} z = 0, \\ -\infty \leq (x, v) \leq +\infty. \end{gathered} \tag{7}$$

The problem given by Eqn. 7 is an example of a general MCP (Dirkse and Ferris 1995a), for which it is required to find $z \in \Re^{n}$ for given lower l and upper bounds u $(-\infty \leq \ell \leq u \leq +\infty)$ and a function $F : \Re^{n} \to \Re^{n}$, such that precisely one of the following holds for each $i \in \{1, \ldots, n\}$:

$$
\begin{aligned}
&F_i = 0 \quad \text{and} \quad \ell_i \le z_i \le u_i, \\
&F_i > 0 \quad \text{and} \quad z_i = \ell_i, \\
&F_i < 0 \quad \text{and} \quad z_i = u_i.
\end{aligned} \tag{8}
$$

Our MCP (Eqn. 7) can be solved using standard methods if hardening is adopted. In certain instances, however, such as when softening laws are assumed, there is no guarantee that any of known algorithms will solve the problem (see e.g. Tin-Loi and Ferris 1997).

Minimum Weight Problem (MPEC)

The minimum weight problem we use as an example of an MPEC was first formulated by Kaneko and Maier (1981) and later revisited by Ferris and Tin-Loi (1999a) with a view towards more efficient and robust solution schemes, especially for large-scale structures. The problem can be described briefly as follows. Under the assumptions of a fixed topology and specified loads, we wish to minimize the volume of the structure under the additional constraints that certain or all displacements and plastic deformations are kept within prescribed serviceability limits. The yield limits *r*, stiffnesses *S* and hardening parameters *H* of the constituent members of the structure are all regarded as unknown but assumed to be (continuous) functions of the cross-sectional areas of all *n* elements.

For simplicity of exposition, assume a truss-like structure of *n* members, for which the unknown element areas are collected in vector $a \in \Re^n$ and the known element lengths in vector $l \in \Re^n$. Assuming that explicit expressions for member stiffnesses $S(a)$ and hardening matrices $H(a)$, in terms of *a* are available, the minimum volume (weight) problem can then be formally stated as the following constrained optimization problem in (a, x, v, z):

$$
\begin{gathered}
\min \quad l^T a \\
\text{subject to} \quad C^T x - f = 0, \\
x - S(a)Cv + S(a)Nz = 0, \\
w \equiv -N^T x + H(a)z + r(a) \ge 0, \quad z \ge 0, \quad w^T z = 0, \\
-\hat{v} \le v \le \hat{v}, \\
z \le \hat{z}, \\
a_\ell \le a \le a_u, \\
Ta = 0,
\end{gathered} \tag{9}
$$

where $\hat{v} \in \Re^d$ is a vector of nonnegative deflections limits; $\hat{z} \in \Re^y$ is a vector of prescribed upper bounds on plastic multipliers used to model the limited ductility of the members; $a_\ell \in \Re^n$ and $a_u \in \Re^n$ are, respectively, lower and upper bounds on the cross-sectional areas;

and $T \in \Re^{t \times n}$ is a technological matrix imposing t constraints (e.g. identical areas for groups of members) on the design variables (areas).

The optimization problem given by Eqn. 9 is a special case of an MPEC (Luo *et al.* 1997) in which the equilibrium system takes the form of a complementarity condition. MPECs are much harder to solve than MCPs. Whilst an extensive theory of first and second order optimality conditions for MPECs has been developed, still relatively little is known about the numerical solution of practical, large-scale MPECs likely to arise in realistic applications. The most prominent feature of an MPEC, and one that distinguishes it from a standard nonlinear program, is the presence of complementarity constraints. These constraints classify this class of mathematical programs as a nonlinear disjunctive (or piecewise) program and therefore carries with it a "combinatorial curse". Recent work by Dirkse and Ferris (1999) describes several new tools for modeling MPECs that are built around the introduction of an MPEC model type into the GAMS language, ready to be linked to newly developed solvers. We, however, have had considerable success in modeling and solving MPECs for a variety of engineering mechanics problems (Ferris and Tin-Loi 1998, 1999a, 1999b) as a series of nonlinear programming problems using the GAMS environment and its associated nonlinear programming solver CONOPT (Drud 1994).

MODELING WITH GAMS

GAMS is a high-level modeling language specially designed to facilitate the construction, solution and maintenance of large and complex mathematical programming models. It is a high level declarative language for formulating small to very large mathematical programming models using simple and concise algebraic statements which mirror the actual mathematical constructs involved. A GAMS model is transparent to both human and computer, is easily modified and moved across different computing platforms from notebooks to mainframes, and is independent of the solution algorithm of the mathematical programming solvers. It not only frees the model builder from the burdens imposed by the solution phase but also takes over the steps required for generation of the model. In addition to providing simplicity and compactness of model construction, it possesses important capabilities such as an internal efficient sparse data representation and automatic differentiation.

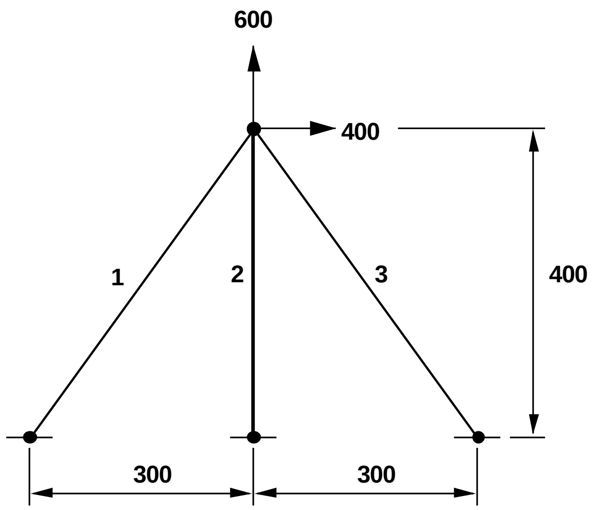
A number of mathematical programming problems types can be solved via GAMS. In addition to the MCP problem type, other available model types are LP (linear programming), NLP (nonlinear programming), MIP (mixed integer programming), RMIP (relaxed mixed integer programming), MINLP (mixed integer nonlinear programming), RMINLP (relaxed mixed integer nonlinear programming) and CNS (constrained nonlinear systems). GAMS is continually evolving and adapted as new algorithms and problem classes have been explored. We refer the interested reader to the extensive GAMS library of models (from such diverse areas as economics, chemical engineering, trade, etc.) accessible from the GAMS website (`http://www.gams.com`), and to Dirkse and Ferris (1995b) for GAMS models of MCPs.

In order to illustrate the typical structure and syntax of a GAMS model, we list in Fig. 1 a simple model (`state.gms`) suitable for the large-scale holonomic analysis of elastoplastic trusses. The GAMS file is written using a standard text editor and executed through a "`gams state`" command. Readers versed in GAMS will recognize the `sets`, `variables`, etc. declarations, while those not familiar with GAMS will appreciate the concise yet descriptive style and will also immediately recognize the parallel to the MCP given by Eqn. 7. We have purposely separated the model proper from its input data which is inserted at compile time through the `$include state.dat` statement. Note also the (optional) matching of free variables to equations in the `model` statement, allowing GAMS to check that there are the same number of free variables as equations.

```
sets
  d         'No. of structure dof'
  m         'No. of members'
  y         'No. of yield functions per member';

alias (y,yy);

parameters
  f(d)      'Load vector'
  C(m,d)    'Compatibility matrix'
  S(m)      'Member stiffness'
  N(m,y)    'Normal matrix'
  H(m,y,y)  'Hardening matrix'
  r(m,y)    'Yield limits';

variables
  x(m)      'Member stresses'
  v(d)      'Displacements';

positive variables
  z(m,y)    'Plastic multipliers';

equations
  eq1(d)
  eq2(m)
  eq3(m,y);

eq1(d)   .. sum(m,C(m,d)*x(m))-f(d) =e= 0;
eq2(m)   .. (1/S(m))*x(m)-sum(d,C(m,d)*v(d))+sum(y,N(m,y)*z(m,y)) =e= 0;
eq3(m,y) .. -N(m,y)*x(m)+sum(yy,H(m,y,yy)*z(m,yy))+r(m,y) =g= 0;

model state /eq1.v, eq2.x, eq3.z/;
$include state.dat
solve state using mcp;
display v.l,x.l,z.l;
```

Figure 1: A simple GAMS MCP state model `state.gms`

As a simple and specific example of data input, consider the academic three-bar truss shown in Fig. 2. Relevant data `state.dat` in GAMS format are also indicated. The compatibility matrix for this simple example is explicitly entered (rather than generated as it would be for a large problem). A simple noninteracting hardening matrix (with nonzero diagonal entries) is assumed. On executing this GAMS model using the default solver PATH a displacement vector v = (1.518, 0.642) and indication that only bar 1 yields (in tension) will be obtained.




```
sets  d    / d1*d2 /
      m    / m1*m3 /
      y    / y1*y2 /;

f("d1") = 400; f("d2") = 600;

C("m1","d1") = 0.6;
C("m1","d2") = 0.8;
C("m2","d2") = 1;
C("m3","d1") = -0.6;
C("m3","d2") = 0.8;

S("m1") = 400; S("m2") = 500;
S("m3") = S("m1");

N(m,"y1") = 1; N(m,"y2") = -1;
H(m,y,y) = 0.125*S(m);
r(m,y) = 500;
```

Figure 2: Three-bar truss and associated `state.dat` input

CONCLUSIONS

A large number of important problems in nonlinear mechanics (e.g. plasticity and those with contact-like conditions) involve complementarity relationships. Indeed, as shown in this paper, the most natural, elegant and powerful method of tackling these problems is often as mathematical programming problems involving the complementarity relations explicitly. The two main problem classes (MCP and MPEC) which arise can both be modeled and solved within the GAMS modeling environment, using its industry standard solvers.

We argue, by illustrating how easily MCPs and MPECs can be formulated and modeled for a specific instance of holonomic plasticity, that modeling systems can provide the impetus required for wider use of mathematical programming methods in solving practical problems involving complementarity conditions in engineering mechanics. Hopefully, such work will also lead to increased synergetic interaction between modelers and algorithm developers.

ACKNOWLEDGMENTS

This work was supported by the Australian Research Council and the National Science Foundation.

REFERENCES

Brooke, A., Kendrick, D. and Meeraus, A. (1992) *GAMS: A User's Guide, Release 2.25*, Boyd & Fraser Publishing Company, Massachusetts.

Cohn M.Z. and Maier, G. (eds.) (1979) *Engineering Plasticity by Mathematical Programming*, NATO Advanced Study Institute, Pergamon Press, New York.

Dirkse, S.P. and Ferris, M.C. (1995a) The PATH solver: a non-monotone stabilization scheme for mixed complementarity problems, *Optimization Methods & Software*, **5**, 123-156.

Dirkse, S.P. and Ferris, M.C. (1995b) MCPLIB: a collection of nonlinear mixed complementarity problems, *Optimization Methods & Software*, **5**, 319-345.

Dirkse, S.P. and Ferris, M.C. (1999) Modeling and solution environments for MPEC: GAMS & MATLAB, In: *Reformulation: Nonsmooth, Piecewise Smooth, Semismooth and Smoothing Methods* (Eds. Fukushima, M. and Qi, L.), Kluwer Academic Publishers, 127-148.

Drud, A. (1994) CONOPT - a large-scale GRG code, *ORSA Journal on Computing*, **6**, 207-216.

Ferris, M.C. and Pang, J.S. (1997a) Engineering and economic applications of complementarity problems, *SIAM Review*, **39**, 669-713.

Ferris, M.C. and Pang, J.S. (eds.) (1997b) *Complementarity and Variational Problems: State of the Art*, SIAM, Philadelphia, Pennsylvania.

Ferris, M.C. and Tin-Loi, F. (1998) Nonlinear programming approach for a class of inverse problems in elastoplasticity, *Structural Engineering and Mechanics*, **6**, 857-870.

Ferris, M.C. and Tin-Loi, F. (1999a) On the solution of a minimum weight elastoplastic problem involving displacement and complementarity constraints, *Computer Methods in Applied Mechanics and Engineering* (in print).

Ferris, M.C. and Tin-Loi, F. (1999b) Limit analysis of frictional block assemblages as a mathematical program with complementarity constraints, Technical Report TR99-01, Computer Sciences Department, University of Wisconsin, Madison.

Kaneko, I. and Maier, G. (1981) Optimum design of plastic structures under displacement constraints, *Computer Methods in Applied Mechanics and Engineering*, **27**, 369-391.

Luo, Z.Q., Pang, J.S. and Ralph, D. (1997) *Mathematical Programs with Equilibrium Constraints*, Cambridge University Press, Cambridge.

Tin-Loi, F. and Ferris, M.C. (1997) Holonomic analysis of quasibrittle fracture with nonlinear softening. In: *Advances in Fracture Research* (Eds. Karihaloo, B.L., Mai, Y.W., Ripley, M.I. and Ritchie, R.O.), Pergamon, Oxford, 2183-2190.