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Absolute Value Equations

Abstract. We investigate existence and nonexistence of solutions for NP-hard equations involving absolute values of variables: $Ax - |x| = b$, where A is an arbitrary $n \times n$ real matrix. By utilizing an equivalence relation to the linear complementarity problem (LCP) we give existence results for this class of absolute value equations (AVEs) as well as a method of solution for special cases. We also give nonexistence results for our AVE using theorems of the alternative and other arguments.

Key words. absolute value equations, linear complementarity problems

AMS Classifications. 90C05, 90C30, 15A06, 15A39

1. Introduction

We consider absolute value equations of the type:

$$Ax - |x| = b, \tag{1}$$

where $A \in R^{n\times n}$, $b \in R^n$ and $|\cdot|$ denotes absolute value. As will be shown, the general linear complementarity problem (LCP) [2,3] which subsumes many mathematical programming problems can be formulated as an absolute value equation (AVE) such as (1). This implies that (1) is NP-hard in its general form. By utilizing this connection with LCPs we are able to give some simple existence results for (1) such as that all singular values of A exceeding 1 implies existence of a unique solution for any right-hand side b. By using theorems of the alternative [4, Chapter 2], we are able to give nonexistence results for (1). We shall also give a method of solution based on successive linear programming.

This work is motivated in part by [7] where a more general AVE,

$$Ax + B|x| = b, \tag{2}$$

is considered with $A \in R^{m\times n}$, $B \in R^{m\times n}$ and $b \in R^m$. By specializing (2) to the important case of (1) we obtain new results in this work.

The significance of the AVE (1) arises from the fact that linear programs, quadratic programs, bimatrix games and other problems can all be reduced to an LCP [2,3] which in turn is equivalent to the AVE (1). Thus our AVE formulation, which is simpler to state than an LCP, subsumes major fundamental problems of mathematical programming.

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We now describe our notation. All vectors will be column vectors unless transposed to a row vector by a prime $'$. The scalar (inner) product of two vectors x and y in the n-dimensional real space R^n will be denoted by $x'y$. Orthogonality $x'y = 0$ will be denoted by $x \perp y$. For $x \in R^n$ the 1-norm will be denoted by $\|x\|_1$ and the 2-norm by $\|x\|$, while $|x|$ will denote the vector with absolute values of each component of x. The notation $A \in R^{m\times n}$ will signify a real $m \times n$ matrix. For such a matrix A' will denote the transpose of A, A_i will denote the i-th row of A, and A_{ij} will denote the ij-th element of A. A vector of ones in a real space of arbitrary dimension will be denoted by e. A vector of zeros in a real space of arbitrary dimension will be denoted by 0. The identity matrix of arbitrary dimension will be denoted by I. For simplicity, the dimensionality of some vectors and matrices will not be explicitly given. For a square matrix $A \in R^{n\times n}$, $eig(A)$ will denote the set of n eigenvalues of A. We shall write $D = diag(\pm 1)$ for a diagonal matrix D each diagonal element of which is ± 1.

2. Absolute Value Problems and Linear Complementarity Problems

We will start by showing that the AVE (1) is in fact equivalent to a bilinear program (an optimization problem with an objective function that is the product of two affine functions) and to a generalized linear complementarity problem. We will then show equivalence to the ordinary LCP.

Proposition 1. AVE $\Longleftrightarrow$ Bilinear Program $\Longleftrightarrow$ Generalized LCP
The AVE (1) is equivalent to the bilinear program:

$$0 = \min_{x\in R^n}\{((A+I)x-b)'((A-I)x-b) \mid \ (A+I)x-b \geq 0,\ (A-I)x-b \geq 0\}, \quad (3)$$

and the generalized LCP:

$$0 \leq (A+I)x - b \ \perp \ (A-I)x - b \geq 0. \quad (4)$$

Proof. It is obvious that (3) and (4) are equivalent. We will show now that (3) is equivalent to the AVE (1). Note that the following equivalence holds:

$$|x| \leq Ax - b \iff -Ax + b \leq x \leq Ax - b, \quad (5)$$

where the right side of the equivalence constitutes the constraints of (3). Hence,

$$|x| = Ax - b \iff ((A+I)x - b)'((A-I)x - b) = 0 \text{ and } |x| \leq Ax - b. \quad (6)$$

Consequently (3) holds if and only if (1) holds.$\diamondsuit$

We establish now, under mild conditions, equivalence of the AVE (1) to the standard LCP:

$$0 \leq z \ \perp \ Mz + q \geq 0, \quad (7)$$

where $M \in R^{n\times n}$ and $q \in R^n$.

Proposition 2. AVE $\iff$ LCP *(i) Under the assumption that* 1 *is not an eigenvalue of* A*, the AVE (1) can be reduced to the following LCP:*

$$0 \leq z \ \perp \ (A+I)(A-I)^{-1}z + q \geq 0, \tag{8}$$

where:

$$\begin{aligned} q &= ((A+I)(A-I)^{-1} - I)b, \\ z &= (A-I)x - b. \end{aligned} \tag{9}$$

(ii) Conversely, if 1 *is not an eigenvalue of* M*, then the LCP (7) can reduced to the AVE:*

$$(M-I)^{-1}(M+I)x - |x| = (M-I)^{-1}q, \tag{10}$$

where

$$x = \tfrac{1}{2}((M-I)z + q). \tag{11}$$

Remark 1. We note that the AVE (10) above, that is equivalent to the LCP (7), is simpler than that given in [7, Equation (4)].

Proof. (i) To prove the first part, start with the generalized LCP (4), which is equivalent to the AVE (1), to obtain (8) as follows. Use $z = (A-I)x - b$ from (9) in the right inequality of (4) to get $z \geq 0$, which is the left inequality of (8). Then from (9) set $x = (A-I)^{-1}(z+b)$ in the left inequality of (4) to get:

$$0 \leq (A+I)(A-I)^{-1}(z+b) - b = (A+I)(A-I)^{-1}z + ((A+I)(A-I)^{-1} - I)b, \tag{12}$$

which gives the right inequality of (8) with q as defined in (9). The orthogonality in (8) follows from that of (4).

(ii) To establish the converse, we again use the generalized LCP (4), which is equivalent to the AVE (1) as follows. Start with the LCP (7) and set the left and right side terms of (4) equal to right and left side terms respectively of (7) as follows:

$$\begin{aligned} (A+I)x - b &= Mz + q, \\ (A-I)x - b &= z. \end{aligned} \tag{13}$$

This results in:

$$\begin{aligned} x &= (A-I)^{-1}(z+b), \\ Mz + q &= M(A-I)x - Mb + q = (A+I)x - b. \end{aligned} \tag{14}$$

To satisfy the last equality of (14) for all $x \in R^n$, set:

$$\begin{aligned} b &= (M-I)^{-1}q, \\ A &= (M-I)^{-1}(M+I). \end{aligned} \tag{15}$$

Substituting from (15) in (14) gives:

$$\begin{aligned} x &= (A-I)^{-1}(z+b) \\ &= ((M-I)^{-1}(M+I) - I)^{-1}(z + (M-I)^{-1}q) \\ &= ((M+I) - (M-I))^{-1}((M-I)z + q) \\ &= \tfrac{1}{2}((M-I)z + q). \end{aligned} \tag{16}$$

Hence AVE (1) holds with A, b as given in (10) and x as defined in (11).$\diamondsuit$

We note that the AVE (1) is NP-hard. This was shown in [7, Proposition 2] by reducing the LCP corresponding to the NP-hard knapsack feasibility problem to an AVE.

We also note that the bilinear formulation (3) can serve as a method of solution for the AVE as was bilinear programming exploited in [1,6] for other problems. However in general, the bilinear program (3) is a nonconvex problem and a solution is not always guaranteed. We will show in Corollary 1 below that under certain assumptions on A, the bilinear program (3) is convex and a solution exists.

We turn now to existence results for the AVE (1).

3. Existence of Solution for AVEs

Our existence results are based on the reduction of the AVE (1) in Proposition 2 to the LCP (8). Existence results are well known for LCPs with various classes of matrices [3]. We first prove a simple lemma.

Lemma 1. *Let $S \in R^{n\times n}$ denote the diagonal matrix constituting the nonnegative singular values of A. Then:*

$$min\ eig(A'A) > (\geq) 1 \iff S > (\geq) I, \tag{17}$$

where min eig denotes the least eigenvalue.

Proof. Let USV' be the singular value decomposition of A, where U and V are orthogonal matrices and S is a nonnegative diagonal matrix of singular values. Then,

$$A = USV',\ A' = VSU',\ A'A = VSU'USV' = VS^2V'. \tag{18}$$

Hence, the diagonal elements of S^2 constitute the set of eigenvalues of $A'A$ and the columns of V the eigenvectors of $A'A$. Hence $S > (\geq) I$, which is is equivalent to $S^2 > (\geq) I$, is in turn equivalent to $eig(A'A) > (\geq) 1$.$\diamondsuit$

We turn now to our existence result.

Proposition 3. Existence of AVE Solution

(i) The AVE (1) is uniquely solvable for any $b \in R^n$ if the singular values of A exceed 1.

(ii) If 1 is not an eigenvalue of A and the singular values of A are merely greater or equal to 1, then the AVE (1) is solvable if the bilinear program (3) is feasible, that is:

$$\{x \mid (A + I)x - b \geq 0,\ (A - I)x - b \geq 0\} \neq \emptyset. \tag{19}$$

Proof. (i) We first show that $(A-I)^{-1}$ exists. For, if not, then for some $x \neq 0$ we have that $(A-I)x = 0$, which gives the contradiction:

$$x'x < x'A'Ax = x'A'x = x'Ax = x'x,$$

where the first inequality follows from $eig(A'A) > 1$ as a consequence of Lemma 1. Hence $(A-I)^{-1}$ exists. It follows by Proposition 2 that the AVE (1) can be reduced to the the LCP (8). We show now that the LCP (8) is uniquely solvable by showing that $(A+I)(A-I)^{-1}$ is positive definite [3, Chapter 3]. Since $eig(A'A) > 1$ it follows that $z'(A'A-I)z > 0$ for $z \neq 0$, which is equivalent to $z'(A'-I)(A+I)z > 0$ for $z \neq 0$. Letting $z = (A-I)^{-1}y$ gives that $y'(A+I)(A-I)^{-1}y > 0$ for $y \neq 0$. Hence $(A+I)(A-I)^{-1}$ is positive definite and the LCP (8) is uniquely solvable for any $q \in R^n$ and so is the AVE (1) for any b.

(ii) We note that the feasibility condition (19) is a necessary condition for the solvability of the AVE (1) because it is equivalent to the condition:

$$\{x \mid Ax - b \geq |x|\} \neq \emptyset. \tag{20}$$

By a similar argument as that of part (i) above we have that the matrix $(A+I)(A-I)^{-1}$ of the corresponding LCP (8) is positive semidefinite and hence (8) is solvable if it is feasible. That it is feasible, follows from the assumption (19). $\diamondsuit$

Corollary 1. Bilinear Program Convexity *Under the assumptions of Proposition 3(ii) the bilinear program (3) is convex.*

Proof. The Hessian of the objective function of (3) is $2(A'A-I)$, which is positive semidefinite by (17). $\diamondsuit$

Remark 2. The bilinear program (3), which is equivalent to AVE (1), can be solved by a finite number of successive linear programs obtained by linearizing its objective function around the current iterate under the assumptions of Proposition 3(ii) [1].

Another interesting existence result is the following.

Proposition 4. Unique Solvability of AVE *The AVE (1) is uniquely solvable for any b if $\|A^{-1}\| < 1$.*

Proof. Let USV' be the singular value decomposition of A. Then:

$$\begin{aligned}\|A\|^2 &= \max_{\|x\|=1} \|Ax\|^2 = \max_{\|x\|=1} x'A'Ax \\ &= \max_{\|x\|=1} x'VSU'USV'x = \max_{\|x\|=1} x'VS^2V'x \\ &= \max_{\|y\|=1} yS^2y = \|S\|^2, \text{ where } x = Vy.\end{aligned}$$

Hence, the assumption that $\|A^{-1}\| < 1$ is equivalent to $S > I$ applies; accordingly by Proposition 3(i) the AVE (1) is uniquely solvable for any b. $\diamondsuit$

Remark 3. We note that much more general existence results can be given by invoking classes of matrices for which the LCP (8) is solvable [3,5]. In fact there is a large class of matrices, the class Q for which the LCP is always solvable for any value of q. Thus whenever $(A+I)(A-I)^{-1} \in Q$ the AVE (1) is solvable for any $b \in R^n$. The class Q includes, for example, strictly copositive matrices M, that is $z'Mz > 0$ for all $z > 0$, which in turn includes positive definite matrices. Another useful class is the class Q_0 for which the LCP (8) is solvable whenever it is feasible, that is whenever the inequalities of (8) are satisfied. This includes the class of copositive-plus matrices, that is $z'Mz \geq 0$ whenever $z \geq 0$ and in addition $(M+M')z = 0$ whenever $z'Mz = 0$. This includes the class of positive semidefinite matrices.

Alternatively, the existence part of Proposition 4 may be established by rewriting AVE in the equivalent form $x = A^{-1}|x| + A^{-1}b$ and establishing the convergence of the iteration $x^{k+1} = A^{-1}|x^k| + A^{-1}b$. Uniqueness of the solution then follows easily from this representation by considering the difference of two solutions and obtaining a contradiction using $\|A^{-1}\| < 1$.

Another simple existence result based on the iteration $x^{k+1} = Ax^k - b$ is the following.

Proposition 5. Existence of Nonnegative Solution *Let $A \geq 0$, $\|A\| < 1$ and $b \leq 0$, then a nonnegative solution to the AVE (1) exists.*

Proof. As a consequence of the iteration $x^{k+1} = Ax^k - b$ with $x^0 = -b$, the iterates $\{x^k\}$ are nonnegative and, since $\|A\| < 1$, converge to a solution $x^* \geq 0$ that satisfies $x^* = Ax^* - b$ and hence $Ax^* - |x^*| = b$. $\diamondsuit$

Another existence result can also be given based on the iteration $x^{k+1} = Ax^k - b$ (with $x^0 = -b$) for the case when $\|A\|_\infty$ (as opposed to $\|A^{-1}\|$) is small. This case corresponds to dominance of the $|x|$ term in AVE (whereas Proposition 4 corresponds to dominance of the linear terms in AVE), and leads to 2^n distinct solutions under suitable assumptions as follows.

Proposition 6. Existence of 2^n Solutions *If $b < 0$ and $\|A\|_\infty < \gamma/2$ where $\gamma = \min_i |b_i| / \max_i |b_i|$, then AVE has exactly 2^n distinct solutions, each of which has no zero components and a different sign pattern.*

Proof. Consider the iteration $x^{k+1} = Ax^k - b$, with starting point $x^0 = -b > 0$. We will demonstrate that this iteration converges to a solution $x^* > 0$, so that $|x^*| = x^*$ and the AVE is satisfied. It is easily verified by induction that $x^{k+1} - x^k = -A^{k+1}b$, and thus standard arguments using $\|A\|_\infty < 1$ establish the convergence of the iteration and also yield

$$\begin{aligned}\|x^* - x^0\|_\infty &\leq \|(x^1 - x^0) + (x^2 - x^1) + \ldots\|_\infty \\ &< (\gamma/2 + (\gamma/2)^2 + \ldots)\|b\|_\infty \\ &\leq \gamma\|b\|_\infty \leq \min_i |b_i|.\end{aligned}$$

It follows that,

$$\|x^* + b\|_\infty < \min_i - b_i,$$

and

$$-x_i^* - b_i \le |x_i^* + b_i| < -\max_i b_i,\ i = 1, \dots n.$$

Consequently:

$$0 \le (\max_i b_i) - b_i < x_i^*,\ i = 1, \dots n.$$

Hence $x^* > 0$.

Note that $\|A\|_\infty < 1$ implies that x^* is the unique solution of the system $Ax - x = b$ and hence x^* is the unique positive solution of AVE. For solutions with other sign patterns, note that for a D with $|D| = I$, $Ax - |x| = b$ has a solution with sign pattern D (i.e., $Dx \ge 0$) iff the system $ADy - y = b$, resulting from the substitution $x = Dy$, has a nonnegative solution y. Since $\|AD\|_\infty = \|A\|_\infty$, we may apply the argument of the preceding paragraph to the matrix AD to demonstrate convergence of this modified linear system with coefficient matrix AD to a solution $y^* > 0$, which is equivalent to the existence of a solution $Dy*$ with sign pattern D to AVE. As before, the system $ADy - y = b$ has a unique solution, so the solution Dy^* is the unique AVE solution with sign pattern D. $\diamondsuit$

We turn now to nonexistence results for AVEs.

4. Nonexistence of Solutions for AVEs

We shall utilize theorems of the alternative [4, Chapter 2] as well as a simple representation (24) of $|x|$ to establish mainly nonexistence results for AVEs here. We note that the significance of these somewhat negative results is that it may otherwise take the solution of 2^n linear equations to determine whether an AVE has no solution. In contrast, the proposed nonexistence results can be checked by solving a single linear program in polynomial time, as in the case of verifying (21), or merely by observation, as in Propositions 9 and 10.

We begin with a simple nonexistence result based on the infeasibility of the feasible region of the bilinear program (3) which is equivalent to te AVE (1).

Proposition 7. Nonexistence of Solution *The AVE (1) has no solution for any A, b such that:*

$$r \ge A'r \ge -r,\ b'r > 0,\ \text{has solution } r \in R^n. \tag{21}$$

Proof. By the Farkas theorem of the alternative [4, Chapter 2] we have that (21) is equivalent to:

$$(A + I)u + (-A + I)v = -b,\ (u, v) \ge 0,\ \text{has no solution } (u, v) \in R^{2n}.$$

Making the transformations, $x = -u+v$, $s = u+v$, or equivalently, $u = (s-x)/2$, $v = (s + x)/2$, results in:

$$-Ax + s = -b,\ s + x \ge 0,\ s - x \ge 0,\ \text{has no solution } (s, x) \in R^{2n}.$$

That is,

$$(A+I)x-b\geq 0,\ (A-I)x-b\geq 0,\ \text{has no solution } x\in R^n.$$

This is equivalent to the feasible region of the bilinear program (3) being empty and hence the AVE (1) has no solution. $\diamondsuit$

We give a simple example of this proposition.

Example 1. Consider the AVE:

$$\begin{bmatrix}-1 & 2\\ 1 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}-\begin{vmatrix}x_1\\ x_2\end{vmatrix}=\begin{bmatrix}-1\\ 1\end{bmatrix}. \tag{22}$$

This AVE has no solution because it can be easily checked that $r=\begin{bmatrix}4\\ 6\end{bmatrix}$ satisfies the nonexistence condition (21).

We note however that condition (21) is *sufficient* but not necessary in general for nonexistence as the following example shows.

Example 2. The AVE:

$$\begin{bmatrix}5 & -2\\ 2 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}-\begin{vmatrix}x_1\\ x_2\end{vmatrix}=\begin{bmatrix}3\\ 2\end{bmatrix}, \tag{23}$$

has no solution. However, (21) has no solution as well for this case.

For the next set of results we shall make use of the simple fact that for $x\in R^n$:

$$|x|=Dx,\ \forall D=diag(\pm 1) \text{ such that } Dx\geq 0. \tag{24}$$

Using this fact we shall first give a couple of existence results and then our final nonexistence result.

Proposition 8. Existence for a Class of AVEs *Let $C\in R^{n\times n}$ and $b\in R^n$. Then:*

$$(C-I)z=b,\ z\geq 0,\ \text{has a solution } z\in R^n, \tag{25}$$

implies that:

$$Ax-|x|=b \text{ has a solution } \forall A=CD,\ D=diag(\pm 1). \tag{26}$$

Remark 4. We note that the assumption (25) can be easily checked by solving a single linear program.

Proof. By setting $z=Dx$, we note that condition (25) is equivalent to the following:

$$\forall D=diag(\pm 1)),\ CDx-Dx=b,\ Dx\geq 0,\ \text{has a solution } x\in R^n. \tag{27}$$

Setting $A=CD$ and making use of (24) gives (26). $\diamondsuit$

We have the following corollary.

Corollary 2. *Under the assumption (25), there exist* 2^n *solvable AVEs (1) where* $A = CD$*,* $D = diag(\pm 1)$ *and* $x = Dz$*.*

We now give two final nonexistence results that are related to Propositions 5 and 6.

Proposition 9. Nonexistence of Solution *Let* $0 \neq b \geq 0$ *and* $\|A\| < 1$*. Then the AVE (1) has no solution.*

Proof. We will show that if the AVE (1) has a nonzero solution then b must contain at least one negative element. Rewriting AVE (1) as $|x| - Ax = -b$, note that the LHS $|x| - Ax$ has at least one positive element when $x \neq 0$. Otherwise, $|x| - Ax \leq 0$, and consequently $|x| \leq Ax$ leads to the contradiction $\|x\| \leq \|Ax\| \leq ||A\| \ \|x\| < \|x\|$. Thus b must contain at least one negative element. $\diamondsuit$

Proposition 10. Nonexistence of Solution *If* b *has at least one positive element and* $\|A\|_\infty < \bar{\gamma}/2$ *where* $\bar{\gamma} = \max\limits_{b_i > 0} b_i / \max\limits_i |b_i|$*, then AVE has no solution.*

Proof. Suppose the conditions hold and AVE has a solution, so that, for some for some diagonal matrix D with $|D| = I$, $Ax - Dx = b$ has a solution with $Dx = |x|$. Thus for $\bar{D}$ with $|\bar{D}| = I$ such that $\bar{D}b = -|b|$ we have that $\bar{D}Ax - \bar{D}Dx = -|b|$ also has a solution x^* with $Dx^* = |x^*|$. Since multiplication by $\bar{D}$ and D have no effect on $\|A\|_\infty$, this implies that $\bar{D}A\bar{D}Dy - y = -|b|$ has a unique solution y^* with $\bar{D}Dx^* = y^*$. Now consider y_j^*, where j corresponds to the largest positive element of b. By the approach of the proof of Proposition 6, it is easily shown that $y_j^* > 0$. However, since $0 < y_j^* = \bar{d}_j d_j x_j^*$, where $\bar{d}_j$ and d_j are the jth diagonal elements of $\bar{D}$ and D, there is a contradiction since $d_j x_j^* > 0$ and $\bar{d}_j < 0$. $\diamondsuit$

5. Conclusion and Outlook

The AVE (1) constitutes one of the most simply stated NP-hard problems. As such it is a fascinating problem to investigate theoretically and computationally. In this work we have established existence and nonexistence results for classes of AVEs and indicated a method of solution when a solution exists for a class of these equations. Further relations with wider classes of linear complementarity and other problems may shed further light and generate new methods of solution and insights into this deceptively simple looking class of NP-hard problems.

Acknowledgements. The research described in this Data Mining Institute Report 05-06, November 2005, revised May 2006, was supported by National Science Foundation Grants CCR-0138308, IIS-0511905 and DMI-0400294.

References

1. K. P. Bennett and O. L. Mangasarian. Bilinear separation of two sets in n-space. *Computational Optimization & Applications*, 2:207–227, 1993.
2. R. W. Cottle and G. Dantzig. Complementary pivot theory of mathematical programming. *Linear Algebra and its Applications*, 1:103–125, 1968.
3. R. W. Cottle, J.-S. Pang, and R. E. Stone. *The Linear Complementarity Problem*. Academic Press, New York, 1992.
4. O. L. Mangasarian. *Nonlinear Programming*. McGraw–Hill, New York, 1969. Reprint: SIAM Classic in Applied Mathematics 10, 1994, Philadelphia.
5. O. L. Mangasarian. Characterization of linear complementarity problems as linear programs. *Mathematical Programming Study*, 7:74–87, 1978.
6. O. L. Mangasarian. The linear complementarity problem as a separable bilinear program. *Journal of Global Optimization*, 6:153–161, 1995.
7. O. L. Mangasarian. Absolute value programming. Technical Report 05-04, Data Mining Institute, Computer Sciences Department, University of Wisconsin, Madison, Wisconsin, September 2005. ftp://ftp.cs.wisc.edu/pub/dmi/tech-reports/05-04.ps. *Computational Optimization and Applications, to appear.*