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Uniqueness of Integer Solution of Linear Equations

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Abstract. We consider the system of m linear equations in n integer variables $Ax = d$ and give sufficient conditions for the uniqueness of its integer solution $x \in \{-1,1\}^n$ by reformulating the problem as a linear program. Uniqueness characterizations of ordinary linear programming solutions are utilized to obtain sufficient uniqueness conditions such as the intersection of the kernel of A and the dual cone of a diagonal matrix of ±1's is the origin in R^n. This generalizes the well known condition that $\ker(A) = 0$ for the uniqueness of a non-integer solution x of $Ax = d$. A zero maximum of a single linear program ensures the uniqueness of a given integer solution of a linear equation.

Keywords: integer solution uniqueness, linear equations, absolute value equations, linear programming

1. INTRODUCTION

We consider the system of linear equations in integer variables:

$$Ax = d, \quad x \in \{-1,1\}^n, \tag{1}$$

where A is a given real matrix in $R^{m\times n}$, $d \in R^m$ and $x \in \{-1,1\}^n$ is an n-dimensional vector of integers each component of which is ±1. This problem can be considered a generalization of the classical knapsack feasibility problem [6, 3, 1, 2] of finding an n-dimensional binary integer vector $y \in \{0,1\}^n$ such that:

$$a^T y = c, \tag{2}$$

where a is an n-dimensional column vector of positive integers and c is a positive integer. An obvious generalization of this is the following generalized multi-knapsack feasibility problem where there are no integrality or nonnegativity restrictions on the $m \times n$ real matrix A or the real vector $b \in R^m$:

$$Ay = b, \ \ y \in \{0,1\}^n. \tag{3}$$

Using the transformation:

$$y = \frac{e-x}{2}, \ x = e - 2y, \tag{4}$$

where e is a column vector of ones, we obtain the absolute value equation [6]:

$$\begin{array}{rcl} |x| &=& e, \\ Ax &=& d, \end{array} \tag{5}$$

where:

$$d = Ae - 2b. \tag{6}$$

It is evident then that (5) equivalent to our original problem (1).

We note here that characterizing uniqueness of solution for integer problems such as (1) is essentially an NP-hard problem [10, 9, 8]. Our interest here is in providing sufficient conditions for uniqueness of a given integer solution to (1), and utilizing these conditions as a computational tool for a class of integer problems.

We now briefly describe the contents of the paper. In Section 2 we present our linear programming formulation (7) and establish uniqueness of an integer solution of (1) via this linear programming formulation. In fact we obtain a sufficient condition for uniqueness of an integer solution of (1) by solving a single linear program (12). In Section 3 we give some computational and uniqueness results for solving (1) via the linear programming formulations of Section 2. Section 4 concludes the paper.

A word about our terminology and notation now. When we refer to an integer solution x of either the linear equation (1) or the linear program (7) below, we mean exactly that $x \in \{-1,1\}^n$ and exclude the case when a component of x is zero. All vectors will be column vectors unless transposed to a row vector by a superscript T. For a vector $x \in R^n$ the notation x_j will signify the j-th component. The scalar (inner) product of two vectors x and y in the n-dimensional real space R^n will be denoted by $x^T y$. For $x \in R^n$, $|x|$ denotes the vector in R^n whose components are the absolute values of the components of x. The notation $A \in R^{m\times n}$ will signify a real $m \times n$ matrix. For such a matrix, A^T will denote the transpose of A, A_i will denote the i-th row and A_{ij} the ijth element. A vector of ones in a real space of arbitrary dimension will be denoted by e. Thus for $e \in R^n$ and $x \in R^n$ the notation $e^T x$ will denote the sum of the components of x. A vector of zeros in a real space of arbitrary dimension will be denoted by 0. The abbreviation "s.t." stands for "subject to".

2. Linear Programming Formulation and Uniqueness of Integer Solution

We begin by stating a rather obvious linear programming relaxation of our integer variable linear equation (1) as follows:

$$\min_x 0^T x \ \ s.t. \ Ax = d, \ \ -e \le x \le e. \tag{7}$$

We note immediately that if this linear program has a unique solution and that solution is integer then by solving this linear program a solution to (1) is obtained. We first state a rather obvious uniqueness result for (7).

Proposition 1 Uniqueness of Solution of Linear Program (7) *The linear program (7) has a unique solution if and only if it gives the same solution for any arbitrary objective function $h^T x$.*

Proof: This is evident or more formally follows from [5, Theorem 1]. ■

This proposition is useful for checking the uniqueness of any solution of the linear program (7), whereas the following proposition establishes the uniqueness of a given solution of (1). Since each solution of (1) is also a solution of the linear program (7), it follows that if a solution of (1) is a unique solution of the linear program (7), then it must be a unique solution of (1).

Proposition 2 Uniqueness of a Given Integer solution of Linear Program (7) and the Linear Equation (1) *Let $x = De$, $D_{ii} = \pm 1,\ i = 1, \ldots, n$, be a solution of (1). Then any one of conditions (i), (ii) or (iii) below, imply that the linear program (7) and hence the linear equation (1) have the unique solution $x = De$.*

(i) The system

$$Ax = 0,\ \ Dx \geq 0,\ \ x \neq 0, \tag{8}$$

has **no** *solution $x \in R^n$.*

(ii) The following holds for the intersection of the kernel of A, $\ker(A)$, and the dual cone $\{x \mid Dx \geq 0\}$ generated by the rows of D:

$$\ker(A) \bigcap \text{dual cone}(D) = 0 \in R^n. \tag{9}$$

(iii) For each $q \in R^n$ the following system has solution $(u, v) \in R^{2n}$:

$$A^T u + Dv = q,\ \ v \geq 0. \tag{10}$$

Proof: We first note that at an integer solution of the linear program (7) we have, for the $n \times n$ diagonal matrix D of ± 1's corresponding to ± 1 components of the integer solution x, that:

$$Ax = d,\ \ Dx = e. \tag{11}$$

Hence the only active inequality constraints of the linear program (7) are those corresponding to $Dx = e$. We then have that the sufficient conditions (i)-(iii) above for the uniqueness of an integer solution of (7) hold because of the following.

(i) This part follows from [5, Theorem 2(iii)].

(ii) This part is essentially is a restatement of (i) this theorem once we note that the dual cone of D is merely the set $\{x \mid Dx \geq 0\}$.

(iii) This part follows from [5, Theorem 2(vii)].

■

An obvious corollary to Proposition 2(i) leads to the following simple linear programming sufficient condition for the uniqueness of a given integer solution of the linear program (7) and hence of our original linear equation (1).

Corollary 3 Uniqueness of Integer Solution of the Linear Program (7) and the Linear Equation (1) *The linear program (7) and hence the linear equation (1) have a unique integer solution $x = De$, where $D_{ii} = \pm 1$, $i = 1, \ldots, n$, if the following linear program has a zero maximum:*

$$\max_x e^T Dx \;\; s.t. \;\; Ax = 0, \;\; Dx \geq 0, \;\; -e \leq x \leq e. \tag{12}$$

Proof: The proof follows directly from Proposition 2(i). ■

We shall now use the linear programming formulation (7) and the results above to generate integer solutions for our problem (1) and determine when its integer solution is unique.

3. Integer Solution of (1) via the Linear Program (7)

We begin the section with two simple examples, the first with a non-unique integer solution and the second with a unique one.

Example 1

$$Ax = d, \text{ where } A = [1 \;\; 1.5 \;\; 1], \;\; d = 1.5. \tag{13}$$

This problem has two integer solutions, $[-1 \;\; 1 \;\; 1]$ and $[1 \;\; 1 \;\; -1]$. If we test for uniqueness of the first solution by solving the linear program (12) we have that:

$$\max_x \{-x_1 + x_2 + x_3 \mid x_1 + 1.5x_2 + x_3 = 0, \; -x_1 \geq 0, \; x_2 \geq 0, \; x_3 \geq 0, \; -e \leq x \leq e\} = 2 > 0, \tag{14}$$

from which we conclude by Corollary 3 that the solution $[-1 \;\; 1 \;\; 1]$ is not necessarily a unique integer solution of (1). A similar result holds if the linear program (12) is applied to the other solution $[1 \;\; 1 \;\; -1]$. Using the linear programming formulation (7) on this example gives the integer solution $[-1 \;\; 1 \;\; 1]$. Changing the objective function coefficient vector of (7) from zero to the integer solution $[-1 \;\; 1 \;\; 1]$ to force (7) away from that solution gives the other integer solution $[1 \;\; 1 \;\; -1]$.

Example 2

$$Ax = d, \text{ where } A = [3 \;\; 2 \;\; -1], \;\; d = 2. \tag{15}$$

By enumeration this problem has a unique integer solution, $[1 \;\; -1 \;\; -1]$. If we test for uniqueness of this integer solution by solving the linear program (12) we have that:

$$\max_x \{x_1 - x_2 - x_3 \mid 3x_1 + 2x_2 - x_3 = 0, \; x_1 \geq 0, \; -x_2 \geq 0, \; -x_3 \geq 0, \; -e \leq x \leq e\} = \frac{7}{3} > 0, \tag{16}$$

from which we conclude by Corollary 3 that the solution $[1 \;\; -1 \;\; -1]$ is not necessarily a unique integer solution of (1). The linear program program (7) obtains the non-integer solution $[-1/3 \;\; 1 \;\; -1]$.

In Table 1 we present computational results for fifteen integer linear equations (1) by solving the linear program (7), utilizing the CPLEX linear programming code [4] within

MATLAB [7]. The integer linear equation (1) was obtained by the transformation (4) of 15 randomly generated solvable multi-knapsack feasibility problems (3) $Ay = b, A \in R^{m\times n}$, $y \in \{0,1\}^n$, with each A_{ij} randomly chosen from the set $\{1,2,\ldots,1000\}$ and such that a solution $y \in \{0,1\}^n$ of (3) exists with approximately a quarter of its components being zero. The times in Table 1 are for a 4 Gigabyte machine running Red Hat Enterprise Linux 5. Column 4 gives the number of non-integer components of a solution of (1) obtained by the the linear program (7). We make the following remarks regarding Table 1:

(i) For 10 of the 15 instances generated, the linear program (7) obtained integer solutions which turned out to be unique solutions of the system of linear equations (1). Uniqueness of an integer solution of (7) and hence of (1) was determined by a zero maximum of the linear program (12) and is signified by a "yes" in column 5 of Table 1.

(ii) In all ten instances for which $m \geq \frac{n}{2}$, the linear program (7) generated an integer solution which turned out to be a unique solution of (1).

(iii) For the five problems for which integer solutions were not obtained by the linear program (7), exactly m non-integer components were generated by the linear program (7) solution for each instance. For these instances, no assertion regarding uniqueness of solution is made in column 5 of Table 1, because the linear program (12) returned a positive maximum for the known integer solution of these problems.

4. Conclusion and Outlook

We have transformed a generalized multi-knapsack feasibility problem into a system of linear equations in ± 1 integer variables and have formulated a linear program (7) for its solution. This linear programming formulation appears to work whenever $m \geq \frac{n}{2}$. We have also given a sufficient conditions for uniqueness of any given integer solution of the system of linear equations (1) by solving a single linear program (12). A topic worth further investigation is that of giving sufficient conditions such as $m \geq \frac{n}{2}$ plus other conditions, if any, that would ensure the generation of an integer solution by the linear program (7).

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Table 1. Integer solution of the system of linear equations (1) $Ax = d$ by the linear program (7), utilizing the CPLEX linear programming code [4] within MATLAB [7]. The times are for a 4 Gigabyte machine running Red Hat Enterprise Linux 5. Column 4 gives the number of non-integer components of a solution obtained by the the linear program (7). Column 5 states whether (1) has a unique solution as determined by the linear program (12) having a zero maximum.

No. of Rows m	No. of Variables n	MATLAB Time Sec toc	No. of Non-Integer Components in LP Solution	Uniqueness of Integer Solution
3	10	0.005	3	—
5	10	0.005	0	yes
8	10	0.004	0	yes
30	100	0.007	30	—
50	100	0.009	0	yes
80	100	0.009	0	yes
150	500	0.131	150	—
250	500	0.266	0	yes
400	500	0.380	0	yes
300	1,000	1.282	300	—
500	1,000	3.308	0	yes
800	1,000	3.391	0	yes
600	2,000	17.499	600	—
1,000	2,000	41.598	0	yes
1,600	2,000	37.777	0	yes

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