A Hybrid Algorithm for Solving the Absolute Value Equation

Olvi L. Mangasarian *

Abstract

We propose a hybrid algorithm for solving the NP-hard absolute value equation (AVE): $Ax-|x|=b$, where A is an $n \times n$ square matrix. The algorithm makes no assumptions on the AVE other than solvability and consists of solving iteratively a linear system of equations followed by a linear program. The algorithm was tested on 100 consecutively generated random solvable instances of the AVE with n =50, 100, 200, 500 and 1,000. The algorithm solved 100% of the test problems to an accuracy of 10^{-8} by solving an average of 2.77 systems of linear equations and linear programs per AVE.

Keywords: absolute value equation, concave minimization, linear programming, linear equations

1 Introduction

We consider the absolute value equation (AVE):

$$Ax - |x| = b, \tag{1}$$

where $A \in R^{n\times n}$ and $b \in R^n$ are given, and $|\cdot|$ denotes absolute value. A slightly more general form of the AVE, $Ax + B|x| = b$ was introduced in [16] that was investigated algorithmically in [17] and in a more general context in [8]. The AVE (1) was investigated in detail theoretically in [13], and a bilinear program in the *primal* space of the problem was prescribed there for the special case when the singular values of A are not less than one. No computational results were given in either [13] or [8]. In contrast in

*Computer Sciences Department, University of Wisconsin, Madison, WI 53706 and Department of Mathematics, University of California at San Diego, La Jolla, CA 92093. *olvi@cs.wisc.edu*.

[7], computational results were given for a linear-programming-based successive linearization algorithm utilizing a concave minimization model, for a primal-dual bilinear model in [11] and for a dual-complementarity based linear programming formulation [12]. As was shown in [13], the general NP-hard linear complementarity problem (LCP) [2, 3, 1], which subsumes many mathematical programming problems, can be formulated as an AVE (1). This implies that (1) is NP-hard in its general form. More recently a generalized Newton method was proposed for solving the AVE (1) [9], while a uniqueness result for the AVE (1) is presented in [18] and for a more general version of the AVE (1) in [19]. Finally, existence and convexity results are given in [4], while an algorithm for computing all solutions of an AVE (1) is given in [20], and equivalent formulations of the AVE (1) are given in [15].

Our present hybrid approach consists of an iterative process whereby we first solve a system of linear equations based on AVE (1) and the $\pm$ signs of the components of the current iterate of x followed by a linear program generated by a concave minimization formulation of AVE (1). This process is repeated until a solution to AVE (1) is obtained. On average this approach generates a solution to the AVE (1) in in less than three iterations. In Section 2 we outline the theory behind our approach and in Section 3 we state our iterative algorithm that consists of iteratively solving a linear system of equations followed by solving a linear program. In Section 4 we give computational results that show the effectiveness of our approach by solving 100% of a sequence of 100 randomly generated consecutive AVEs in R^{50} to $R^{1,000}$ to an accuracy of 10^{-8} . In contrast only 90.2% of the 500 AVEs attempted in [10] were solved by utilizing a linear programming approach only that is proposed there. Similarly only 95% of the problems using merely a linear programming iterative process in [7]. Section 5 concludes the paper.

We describe our notation now. All vectors will be column vectors unless transposed to a row vector by a prime $'$. For a vector $x \in R^n$ the notation x_j will signify the j-th component. The scalar (inner) product of two vectors x and y in the n-dimensional real space R^n will be denoted by $x'y$. For $x \in R^n$, $\|x\|_\infty$ will denote the ∞-norm: $\max\limits_{i=1,\ldots,n} |x_i|$. The notation $A \in R^{m\times n}$ will signify a real $m \times n$ matrix. For such a matrix, A' will denote the transpose of A. A vector of ones in a real space of arbitrary dimension will be denoted by e, while I will denote the identity matrix. Thus for $e \in R^m$ and $y \in R^m$ the notation $e'y$ will denote the sum of the components of y. A vector of zeros in a real space of arbitrary dimension will be denoted by 0. The abbreviation “s.t.” stands for “subject to”.

2 AVE as a Concave Minimization Problem

We begin with the following convex polyhedral region in R^{2n} that contains a solution to AVE (1):

$$Z = \{(x, y) \in R^{2n} \mid y \geq |x|,\ y \geq Ax - b\}. \tag{2}$$

That Z is a polyhedral set follows from its equivalent definition as:

$$Z = \{(x, y) \in R^{2n} \mid y \geq x \geq -y,\ y \geq Ax - b\}. \tag{3}$$

That Z contains a solution of AVE (1) follows by setting:

$$y = |x|, \quad y = Ax - b, \tag{4}$$

which generates a point (x, y) in Z that solves AVE (1). This latter condition (4) can be achieved by the following concave minimization problem having a zero minimum:

$$\{\min_{x,y} e'(y - |x|) + e'(y - Ax + b) \ \text{ s.t. } \ y \geq x \geq -y,\ y \geq Ax - b\} = 0. \tag{5}$$

Our proposed hybrid approach consists of alternating between solving a linear programming linearization of the concave minimization problem (5) to generate a solution (x, y) and using the $\pm$ signs of the solution x to generate an $n \times n$ diagonal matrix E of ± 1 to solve the following linear system of equations:

$$Ax - Ex = b. \tag{6}$$

This turns out to be a very effective and efficient finite hybrid method, not tried before, for solving AVE (1) as we shall describe in the following sections of this paper. We further note that neither an iterative linearization of the concave minimization problem (5) by itself [7] nor an iterative solution of the linear system of equations (6) by itself lead always to an exact solution of AVE (1). That is the principal reason for combining these two approaches into the proposed hybrid approach here.

3 Finite Hybrid Method for Solving AVE

We begin by stating our hybrid linear equation-linear programming algorithm as follows.

Algorithm 3.1 *Choose and accuracy tolerance (typically tol=*10^{-8}*), and maximum number of iterations itmax (typically itmax=* 10*). Initialize the algorithm by setting iteration number* $i = 0$ *and* $x^0 = 0$*.*

(I) *Let* $E = diag(sign(x^i))$ *and solve the linear equation (6) and call its solution* z^i*.*

(II) *Linearize the concave minimization problem (5) around* z^i *as follows:*

$$\begin{array}{llrcl} \min\limits_{x,y} & -e'Ax - sign(z^i)'x + 2e'y & & & \\ \text{s.t.} & & Ax - y & \leq & b, \\ & & x - y & \leq & 0, \\ & & -x - y & \leq & 0. \end{array} \tag{7}$$

Call the solution of this linear program (x^{i+1}, y^{i+1})*.*

(II) *If the number of components satisfying* $|Ax^{i+1} - |x^{i+1}| - b| > tol$ *is* 0 *or if* $i = itmax$ *stop.*

(III) *Set* $i = i + 1$ *and go to Step (I).*

We note that since the feasible region of (7) has a finite number of vertices, then for an infinite sequence $\{(x^i, y^i)\}$ in the above algorithm, one such vertex $(\bar{x}, \bar{y})$ of (7) must appear an infinite number of times if $itmax = \infty$. This fact in itself does not necessarily ensure that $\bar{x}$ is a solution of AVE (1) or that Algorithm 3.1 will terminate at a solution of AVE (1). However, computationally this appears to be case in all cases run, as demonstrated in the next section where exact solutions of AVE (1) are obtained for 100% of 100 randomly generated problems to an accuracy of 10^{-8}.

We further note that since we are linearizing the concave minimization problem (5) in Step (II) of Algorithm 3.1, we can invoke Theorem 3 of [6] which states that for a concave function bounded below on a polyhedral set, successive minimization of the linearized concave function ends in a finite number of steps at a stationary point. This then leads to the following proposition under a stated assumption that takes into account the hybrid feature (I) of Algorithm 3.1.

Proposition 3.2 Finite Termination Result *Under the assumption that* $sign(z^i)'x^{i+1} = sign(x^i)'x^{i+1}$*, Algorithm 3.1 terminates in a finite number of iterations at a stationary point of the concave minimization problem (5) whose global solution solves AVE (1).*

Although this proposition does not guarantee that Algorithm 3.1 terminates at an exact solution of AVE (1), our computational results strongly indicate that it does so in all of the 100 test problems attempted.

4 Computational Results

We implemented our algorithm by solving 100 solvable random instances of the absolute value equation (1) consecutively generated. Elements of the matrix A were random numbers picked from a uniform distribution in the interval $[-5, 5]$. A random solution x with random components from [-.5,.5] was generated and the right hand side b was computed as $b = Ax - |x|$. All computation was performed on 4 Gigabyte machine with a 3159MHz CPU running amd64-rhe16 Linux. We utilized the CPLEX linear programming code [5] within MATLAB [14] to solve our linear programs.

Of the 100 test problems, 100% were solved exactly to an ∞-norm tolerance of 10^{-8}. The maximum number of iterations was set at 10 for $n = 50, 100, 200, 500, 1000$. The computational results are summarized in Table 1 and are better in the percentage of problems solved accurately and in total solution time than those given in [11] for a similar set of problems where a bilinear minimization approach was used but only 88.6% of the problems were solved to a lower accuracy of 10^{-6}. The present results are also better than those of [10] where a different complementarity-based method was utilized to solve only 90.2% of the problems attempted. Furthermore, the present approach is much more general than that of [9] which is applicable only to AVEs with singular values of A greater than 1.

Problem Size n	Number of Problems	Average # of Violated Components of AVE (1) at Initial Iteration Point x^1	Average # of Iterations per Problem	Time in Seconds per Iteration
50	20	0.95	2.15	0.01
100	20	1.45	2.65	0.07
200	20	1.7	3.15	0.24
500	20	1.65	2.70	2.82
1,000	20	3.2	3.2	104.42

Table 1: Computational Results for 100 Random Consecutively Generated AVEs Each Solved to an Accuracy of 10^{-8}.

5 Conclusion and Outlook

We have proposed a hybrid linear equations-linear programming formulation for solving the NP-hard absolute value equation. The method consists of solving a succession of linear equations-linear programs. In 100% of 100 consecutive instances of solvable random test problems, the proposed algorithm solved the problem to an accuracy of 10^{-8}. Possible future work may consist of precise sufficient conditions under which the proposed formulation and solution method is guaranteed to terminate in a finite number of steps.

Acknowledgments

The research described here, based on Data Mining Institute Report 14-02, April 2014, was supported by the Microsoft Corporation and ExxonMobil.

References

[1] S.-J. Chung. NP-completeness of the linear complementarity problem. *Journal of Optimization Theory and Applications*, 60:393–399, 1989.

[2] R. W. Cottle and G. Dantzig. Complementary pivot theory of mathematical programming. *Linear Algebra and its Applications*, 1:103–125, 1968.

[3] R. W. Cottle, J.-S. Pang, and R. E. Stone. *The Linear Complementarity Problem.* Academic Press, New York, 1992.

[4] Sheng-Long Hu and Zheng-Hai Huang. A note on absolute equations. *Optimization Letters*, 4(3):417–424, 2010.

[5] ILOG, Incline Village, Nevada. *ILOG CPLEX 9.0 User's Manual*, 2003. http://www.ilog.com/products/cplex/.

[6] O. L. Mangasarian. Solution of general linear complementarity problems via nondifferentiable concave minimization. *Acta Mathematica Vietnamica*, 22(1):199–205, 1997. ftp://ftp.cs.wisc.edu/math-prog/tech-reports/96-10.ps.

[7] O. L. Mangasarian. Absolute value equation solution via concave minimization. *Optimization Letters*, 1(1):3–8, 2007. ftp://ftp.cs.wisc.edu/pub/dmi/tech-reports/06-02.pdf.

[8] O. L. Mangasarian. Absolute value programming. *Computational Optimization and Applications*, 36(1):43–53, 2007. ftp://ftp.cs.wisc.edu/pub/dmi/tech-reports/05-04.ps.

[9] O. L. Mangasarian. A generalized Newton method for absolute value equations. Technical Report 08-01, Data Mining Institute, Computer Sciences Department, University of Wisconsin, May 2008. ftp://ftp.cs.wisc.edu/pub/dmi/tech-reports/08-01.pdf. Optimization Letters 3(1), January 2009, 101-108. Online version: http://www.springerlink.com/content/c076875254r7tn38/.

[10] O. L. Mangasarian. Absolute value equation solution via dual complementarity. Technical report, Data Mining Institute, Computer Sciences Department, University of Wisconsin, Madison, Wisconsin, September 2011. ftp://ftp.cs.wisc.edu/pub/dmi/tech-reports/11-03.pdf. Optimization Letters 7(4), 2013, 625-630.

[11] O. L. Mangasarian. Primal-dual bilinear programming solution of the absolute value equation. Technical report, Data Mining Institute, Computer Sciences Department, University of Wisconsin, Madison, Wisconsin, February 2011. ftp://ftp.cs.wisc.edu/pub/dmi/tech-reports/11-01.pdf, Optimization Letters 6(7), 1527-1533, 2012.

[12] O. L. Mangasarian. Absolute value equation solution via linear programming. Technical Report 13-01, Data Mining Institute, Computer Sciences Department, University of Wisconsin, Madison, Wisconsin, February 2013. ftp://ftp.cs.wisc.edu/pub/dmi/tech-reports/13-01.pdf. Journal of Optimization Theory and Applications 161, 870-876, 2014.

[13] O. L. Mangasarian and R. R. Meyer. Absolute value equations. *Linear Algebra and Its Applications*, 419:359–367, 2006. ftp://ftp.cs.wisc.edu/pub/dmi/tech-reports/05-06.pdf.

[14] MATLAB. *User's Guide*. The MathWorks, Inc., Natick, MA 01760, 1994-2006. http://www.mathworks.com.

[15] O. A. Prokopyev. On equivalent reformulations for absolute value equations. *Computational Optimization and Applications*, 44(3):363–372, 2009.

[16] J. Rohn. A theorem of the alternatives for the equation $Ax + B|x| = b$. *Linear and Multilinear Algebra*, 52(6):421–426, 2004. http://www.cs.cas.cz/ rohn/publist/alternatives.pdf.

[17] J. Rohn. An algorithm for solving the absolute value equation. *Electronic Journal of Linear Algebra*, 18:589–599, 2009.

[18] J. Rohn. On unique solvability of the absolute value equation. *Optimization Letters*, 3:603–606, 2009.

[19] J. Rohn. A residual existence theorem for linear equations. *Optimization Letters*, 4(2):287–292, 2010.

[20] J. Rohn. An algorithm for computing all solutions of an absolute value equation. *Optimization Letters*, 6:851-856, 2012.