Chapter 1
INTRODUCTION

This dissertation develops general forward-backward splitting methods and their convergence theory for monotone operators. It also covers some algorithm and problem structure analyses and applications to large-scale convex optimaization (extended linear-quadratic programming, optimal control, stochastic programming, etc.), the MPC controller design in chemical engineering, variational inequalities, and so on.

Optimization is a relatively new and fast-developing field of applied mathematics. It requires many subjects of mathematics like functional analysis, real analysis, numerical analysis, differential equations, linear algebra, computational mathematics, convex analysis, nonsmooth analysis as theoretical bases and analytical tools. Its applications cover almost all fields such as physics, chemistry, engineering, industrial management, business, and so on. Simply speaking, optimization deals with problems of maximizing or minimizing some function over some set. Large-scale optimization considers problems with relatively large number of variables and/or constraints which will need too much computer storage space such that regular computers will take relatively long time to solve or can not solve them. The existing methods for large-scale optimization problems are mainly decomposition type. But these decomposition methods have not utilized the special structure of those large-scale problems considered in this dissertation. I use existing splitting algorithms and Lagrangian functions to construct a new kind of primal-dual decomposition methods and it seems that the splitting method which can utilize the structure of our large-scale problems is forward-backward splitting type method. For the convenience of discussion, I consider the convex large-scale problems.

Large-scale convex optimization includes a large class of optimization problems such as problems of quadratic programming, variational inequality, extended linear-quadratic programming, multistage convex programming, discrete-time optimal control, and stochastic programming. Solving large-scale convex problems is generally difficult, especially those involving linear dynamic systems or stochastic variables. In general, those problems can be transformed into constrained large-scale convex saddle point problems by duality theory. This dissertation then concentrates on the nonlinear large-scale saddle point problems related to dynamic systems and stochastic variables. Various other types of problems would be treated as well.

A basic approach for solving a large-scale problem is to use techniques of *decomposition*, i.e., dividing a problem into many smaller ones that can be solved in parallel or in sequence. Conventional mathematical programming decomposition methods, such as Dantzig-Wolfe decomposition (Dantzig 1963) and Benders' decomposition (Benders 1962), typically require specialized block-angular or dual block-angular constraint structures. This generally lead to a relatively coarse-grained partition of the original problem. They are only effective for the problems with small primal or dual problems. This case is not true for those involving dynamic systems, or stochastic variables, which are generally very large for both primal and dual problems (see Rockafellar 1987, 1988, 1990, Rockafellar and Wets 1986, 1990, etc.).

A second method of solving a large-scale problem is to use interior point methods. This is originally from Karmarkar (1984). However, these methods were specially designed for linear programming and quadratic programming (Anstreicher 1991; Lustig, Marsten and Shanno 1991; Mizuno, Todd and Ye 1993; Ji and Ye 1994; Mizuno, Todd and Tuncel 1994; Zhang and Tapia 1993; etc.). This method does not consider the special structure of problems involving dynamic systems or stochastic variables although some (e.g., Wright 1993) have applied the method to special cases of those problems.

The third method of solving a large-scale problem is to use finite envelope techniques (Rockafellar and Wets 1986, 1987, Zhu and Rockafellar 1990). This does use the decomposability of the problems. However, this technique uses a different decomposition than the method in this research. They do not decompose a problem at each iteration into a set of subproblems which could be solved simultaneously.

The goal of this dissertation is to expand a methodology for solving large-scale convex saddle point problems. This method involves dynamic systems, or stochastic variables, by utilizing the monotone operator splitting concept to develop algorithms which are specifically fit to the problem structure. The algorithm will transform a large-scale convex problem, at each iteration, into a set of much smaller subproblems (for instance, all subproblem are one-dimensional for the box-diagonal problems of extended linear-quadratic programming.) that can be solved parallelly. The principle is one of splitting a monotone operator by expressing it as the sum of two simpler operators and then exploiting the resulting structure after analyzing the structure of problems, the suggested title of this splitting method is *"forward-backward splitting"*. This methodology has the advantage of easy (or parallel) implementation and good decomposition.

The topic of monotone operators has a significant history. It was developed originally from functional analysis not optimization. The original definition of a monotone operator was formulated by Kachurovskii (1960). Important early contributions and surveys may be found in studies by Minty (1961, 1962, 1964), Browder (1964, 1965, 1968), Rockafellar (1966, 1967, 1969, 1970), Kachurovskii (1968), etc.. Later, studies in 1974 by Kenderov proved that monotone operators are single-valued almost everywhere. For later development of monotone operators, please refer to publications by Brezis (1973), Vainberg (1973), Pascali and Sburlan (1978), Dolezal (1979), Cesari (1983), Rockafellar (1984), Deimling (1985), Joshi and Bose (1985), Zeidler (1985), and the dissertation by Eckstein (1989).

Many key results regarding the role of monotone operators in optimization are due to Rockafellar (1966, 1967, 1969, 1970, 1976, 1977, 1981, 1984, 1985). Prof. Rockafellar (1970) established the conditions for the maximal monotonicity of subdifferential mappings and the maximality of sums of operators. The general *proximal point algorithm* was codified by Rockafellar (1976). The codification is the base and critical component of the general operator splitting algorithms for finding a zero of a maximal monotone operator, and building on specialized versions devised by Martinet (1970, 1972). Prof. Rockafellar also transformed a convex saddle point problem into one for finding a zero point of a monotone operator in 1970. This is the basic property for this proposed research.

The early operator splitting was well-known in the fields of numerical linear algebra and discretized differential equations. In these fields operators are always linear and single-valued. Please refer to Agmon (1954), Motzkin and Schoenberg (1954), Peaceman and Rachford (1955), Douglas and Rachford (1956), and Douglas and Gunn (1964). The gradient projection method for convex optimization by Goldstein (1964), Levitin and Polyak (1966) is also a special case of operator splitting methods. There are numerous notable species of operator splitting methods. For complete general maximal monotone operators, the development of splitting methods is due to Lions (1978) for the *double backward* scheme, Lions and Mercier (1979) for the *Douglas-Rachford* scheme and the *Peaceman-Rachford* scheme, Passty (1979) for the *forward-backward* scheme, and Spingarn (1983) for the *partial inverse* method.

The forward-backward splitting methods generalize the gradient projection method by Goldstein (1964), Levitin and Polyak (1966). Further special cases of these methods have been described or examined by Bruck (1975, 1977), Polyak (1978), etc. Later, Passty (1979), Gabay (1983), Tseng (1990, 1991), Luo and Tseng (1991, 1992), Chen and Rockafellar (1992) have considered the methods and their applications of this class more generally. Cohen (1980) generalized the gradient projection method to a method called the *auxiliary problem principle.* Dafermos (1983) and Noor (1992) also generalized the gradient projection method and their algorithms could be special cases of the algorithm of Cohen (1988) from Patriksson (1993b). The articles by Cohen (1977, 1980, 1987, 1988), Cohen and Balducchi (1980), Cohen and Joalland (1976), Cohen and Culioli (1987), Cohen and Chaplais (1988), Noor (1992), Dafermos (1990), include applications of their algorithms, which are related to my work in Chapter 5, in variational inequalities, decentralized optimal control, nash equilibria, exchange price equilibria, nondifferentiable optimization, stochastic optimization, and large dynamic constrained optimization.

In his PhD dissertation, Patriksson (1993b) presents a unified framework of iterative algorithms of which the above mentioned forward-backward splitting methods and the algorithms of Cohen, Noor, Dafermos, are special cases. Patriksson gives a convergence analysis of the general unified framework (called the *cost approximation algorithm*), to which our work is related in this dissertation, when applied to problems of nonlinear programming, nondifferentiable optimization and variational inequalities (also refer to Patriksson 1994a-c, 1993a; Larsson and Patriksson 1994). Patriksson introduces line searches in his algorithm and shows that the above algorithms of Cohen, Noor, Dafermos may be described as forward-backward splitting algorithms (possibly combined with line searches). According to this characterization, the convergence analyses of the previous methods and algorithms of forward-backward splitting, especially linear convergence, most often utilize the strong monotonicity of the operator in the Euler (forward) step, not on the backward (proximal point) step. This fact limits the applications of those convergence theorems since the operator in the forward step is only monotone for natural splittings of some large-scale applications, such as those we shall see in the multistage models of Chapter 3 and Chapter 7. One of the key contributions in this dissertation shows linear convergence, which is applicable for the basic (natural) splitting in Chapter 3 and Chapter 7, for general abstract forward-backward splitting algorithms in a Hilbert space without the strong monotonicity of the operator (we call it T_1) in the forward step.

Before we summarize the dissertation, we introduce several concepts.

Let $\mathcal{H}$ always be a real Hilbert space. A *multifunction* $T : \mathcal{H} \rightrightarrows \mathcal{H}$ is a mapping from $\mathcal{H}$ into itself such that $T(w)$ is a subset (possibly empty) of $\mathcal{H}$ for every $w \in \mathcal{H}$. We shall call T an *operator* on $\mathcal{H}$, or over $\mathcal{H}$, in our context.

The *graph* of an operator T is the set

$$\operatorname{gph} T = \{(x, y) \in \mathcal{H} \times \mathcal{H} \,\big|\, y \in T(x), x \in \mathcal{H}\}. \tag{1.1}$$

Definition 1.1. *Let $\langle .,. \rangle$ be the inner product function in $\mathcal{H}$. A multifunction $T : \mathcal{H} \rightrightarrows \mathcal{H}$ is called a monotone operator if*

$$\langle x - x', y - y' \rangle \geq 0 \ \ \textit{whenever} \ \ y \in T(x) \ \textit{and} \ y' \in T(x'). \tag{1.2}$$

It is said to be maximal monotone if, in addition, the graph $\operatorname{gph} T$ is not properly contained in the graph $\operatorname{gph} T'$ of any other monotone operator $T' : \mathcal{H} \rightrightarrows \mathcal{H}$.

The following are some examples of monotone operators: the normal cone operator N_C of a convex set C, the subgradient operator $\partial f : \mathcal{H} \rightrightarrows \mathcal{H}$ of a convex function $f : \mathcal{H} \to \overline{\mathrm{IR}} = [-\infty, \infty]$, any constant operators on $\mathcal{H}$, and positive semidefinite matrices $A : \mathrm{IR}^n \to \mathrm{IR}^n$.

The problem which we shall describe as a *fundamental problem* is

$$(\mathcal{P}_0) \qquad \text{Find a } \; z \in \mathcal{H} \; \text{ such that } \; 0 \in T(z), \quad T - \text{maximal monotone.}$$

Later we shall call any solution of the fundamental problem as a *zero point* of T and we define

$$\mathrm{zer}(T) \equiv T^{-1}(0) := \{z \in \mathcal{H} \,\big|\, 0 \in T(z)\}. \tag{1.3}$$

Some examples for the fundamental problem will be given in Section 2.3. However, the important examples of Problem $(\mathcal{P}_0)$, in this dissertation, are those related to problems, with dynamic systems or stochastic variables, of extended linear-quadratic programming, which will be discussed and described in Chapter 3, Chapter 7, Chapter 8, and Chapter 9.

The basic forward-backward splitting scheme for Problem $(\mathcal{P}_0)$, given $T = T_1 + T_2$ in the study on analysis of Passty, is

$$z^{k+1} \in (I + \lambda_k T_2)^{-1}(I - \lambda_k T_1)(z^k), \qquad \forall \quad k \geq 0. \tag{1.4}$$

Where I is the identity mapping in $\mathcal{H}$ and λ_k are scalars.

The following outline maps the perimeter of this dissertation:

The dissertation consists of four large parts: *review, theory, applications and references.*

PART (I): Review

Chapter 1: In the first chapter, literature review and introduction for the dissertation are given.

Chapter 2: In this chapter, review some related basic concepts, examples, properties, and methods for finding a zero point of a monotone operator such as the Euler method, the proximal point method, and operator splitting methods, as well as their main convergence results.

PART (II): Theory

Chapter 3: Focus on two groups of problems, in Chapter 3, by description and analysis of their inherent structures. These groups can be large scale. Inherent in the structure is a component that is fully decomposable while another part is not. The constraints consist of decomposable boxes and dynamic systems. In addition, four splitting strategies have been applied to these problems. The result of this analysis seems to the conclusion that the forward-backward splitting (FBS) method can take many advantages of the structure and the parallel computing potentials of the problems, compared with other splitting methods. So we suggest the FBS method for this group of problems.

Chapter 4: Since we suggest using mainly the forward-backward splitting (FBS) method, a further analysis for the existing basic forward-backward splitting method and its convergence has been done in Chapter 4. This is due to the fact that the known convergence results of the basic FBS method mainly depend on the strong monotonicity of T_1 or T_1^{-1}. It is not the case for T_1 sometimes. Several new global convergence theorems, that can be always applied to our problems in Chapter 3, are proved and the related convergence rates are exploited for the basic FBS scheme under reasonable assumptions. A superlinear convergence nature for the basic FBS method from (1.4) is deduced in the last section. From the opinion of application, the following relaxation is also generalized and investigated by the author:

$$z^{k+1} \in (I + \lambda_k T_2)^{-1}(I - \lambda_k T_1)(z^k) + \varepsilon_k \mathbb{B}, \quad \text{where} \quad \sum_{k=0}^{\infty} \varepsilon_k < \infty \tag{1.5}$$

and $\mathbb{B}$ is the closed unit ball in $\mathcal{H}$. A number of global and local convergence theorems with convergence rates for the general FBS method generated from (1.5) are developed. The relationship of the convergence conditions and the nonexpansivity of the forword-backward splitting operators $G_k := (I+\lambda_k T_2)^{-1}(I-\lambda_k T_1)$ is explored. The conclusion is that all of the main known or developed convergence conditions are such that G_k for $k = 0, 1, \cdots$ contractive or nonexpansive. The relationship of these convergence conditions is also analyzed. Some equivalent conditions for the convergence conditions on T_1 and T_2 are obtained through the nonexpansivity and the contractivity of operators.

One of the key results in this chapter are the following:

Theorem 1 (ref. Thm. 4.2.2). *If T_1 is Lipschitz continuous, with constant M, and T_2 is strongly monotone, with modulus α, then $\{z^k\}$, from (1.4), strongly converges to the unique zero point of $T = T_1 + T_2$ for any starting point $z^0 \in$ $\mathrm{dom}(T_1)$, when $\lambda_k = \lambda \in \mathbb{R}$ satisfies*

$$0 < \lambda < \lambda^* := \begin{cases} \frac{2\alpha}{M^2-\alpha^2}, & \text{for } M > \alpha; \\ \infty, & \text{for } M \le \alpha \end{cases} \tag{1.6}$$

and the convergence rate will be Q-linear with the global ratio θ. Where

$$\theta^2 = \frac{1 + M^2\lambda^2}{(1+\alpha\lambda)^2} \in (0, 1).$$

We prove a similar conclusion for varied λ_k in Section 4.2. Significance of Theorem 1 is: it does not require T_1 or the inverse of T_1 to be strongly monotone or strictly monotone, and hence is applicable to the saddle-point problems of extended linear-quadratic programming in Chapter 3, Chapter 7, Chapter 8, and Chapter 9, with keeping the good parallel computing structure.

In Section 4.3, several convergence conditions for the forward-backward splitting methods are summarized, of which two are newly developed in this dissertation. They are as follows:

Condition A.

(1) T_2 is strongly monotone with modulus α,

(2) T_1 is Lipschitz continuous with constant M,

(3) $\{\lambda_k\} \subset \mathbb{R}$ satisfies (1.6).

Condition B.

(1) T_1 is Lipschitz continuous with constant M,

(2) T_1, T_2 are strongly monotone with moduli α_1, α_2 respectively,

(3) if for any given λ^- and λ^+,

$$0 < \lambda^- \leq \lambda_k \leq \lambda^+ < \lambda^* := \begin{cases} \frac{2(\alpha_1+\alpha_2)}{M^2-\alpha_2^2}, & \text{for } M > \alpha_2; \\ \infty, & \text{for } M \leq \alpha_2. \end{cases}$$

The theorems in Section 4.3 show the relationship between the convergence conditions and the concept of nonexpansivity of operators. Hence, we can unify the different convergence conditions of the forward-backward splitting methods to a same concept and can analyze the methods uniformly, or can develop new convergence results through the unified framework.

Chapter 5: In Chapter 5, the FBS method is expanded to the extended forward-backward splitting (EFBS) method by the author based on the discussion in Chapter 4. At each iteration, the EFBS method use the following scheme to get new iteration point:

$$z^{k+1} \in (H_k + \lambda_k T_2^k)^{-1}(H_k - \lambda_k T_1^k)(z^k) \tag{1.7}$$

with $\text{zer}(T_1^k + T_2^k) \to \text{zer}(T)$ (see the definition in Section 5.1) as $k \to \infty$, where H_k for $k = 0, 1, \cdots$ are strictly monotone operators in general. This EFBS method is more general and has more flexibility than the FBS method. In this chapter, several convergence theorems for an iteration method generated through the following scheme (1.8) are developed and some convergence rates are analyzed:

$$z^{k+1} \in R_k^{-1} S_k R_k(z^k), \qquad k = 0, 1, \cdots, \tag{1.8}$$

here $R_k : \mathcal{H} \to \mathcal{H}$ is an invertible bounded linear operator and $S_k : \mathcal{H} \rightrightarrows \mathcal{H}$ is an operator for each k; since (1.7) is a special case of (1.8), a number of global and local convergence theorems for the EFBS method are deduced; a local quadratic convergence theorem, two finite convergence theorems and a couple of global superlinear convergence theorems are formed and proved, while the best convergence rate we have so far for the FBS method is Q-linear; the relationship of the EFBS method and several other methods are also explored.

Some major results are summarized as follows:

Theorem 2. *Suppose $\{S_k\}$ is uniformly contractive with constant c and $R_k \to R'$, $R_k^{-1} \to (R')^{-1}$ with R' invertible. If $\mathrm{Fix}(R_k^{-1} S_k R_k) \equiv \mathrm{zer}(T)$, then $\{z^k\}$ from (1.8) strongly converges to the zero of T. Furthermore, convergence rate will be Q-linear with global ratio c when $R_k \equiv R'$.*

Theorem 3. *Suppose $R_k \equiv R$ given R invertible and $\{S_k\}$ is a uniformly contractive sequence with constant c in (5.3.7). Let $\{z^k\}$ be a sequence generated from (1.8). If*

$$\lim_{k\to\infty} \bar{w}_k = w^* \qquad \text{given} \quad \bar{w}_k = S_k(\bar{w}_k) \quad k = 0, 1, \ldots,$$

then $z^k \to R^{-1} w^*$. Moreover, if there is a $\eta < 1$ such that

$$\|\bar{w}_k - w^*\| \le \eta^k C_0 \qquad \text{for any} \quad k \ge 0,$$

where C_0 is a constant, then the convergence rate is linear.

From the above two linear convergence theorems for the unified scheme (1.8), different convergence theorems can be simply deduced or extended for (1.7) depending on conditions on T_1^k, T_2^k, H_k, and λ_k.

Theorem 4 (Quadratic rate). *Suppose $T = T_1^k + T_2^k$ with $\{T_1^k\}$ uniformly strongly monotone with α and uniformly Lipschitz continuous with M. For given $\lambda_k \equiv 1$, if T_1^k is twice differentiable and $H_k = \nabla T_1^k(z^k)$, then the $\{z^k\}$ from (1.7) will converge to the zero point z^* of T at square rate when z^0 is near z^*.*

Theorem 5 (Superlinear rate). *Suppose $T = T_1^k + T_2^k$ with $\{T_1^k\}$ uniformly strongly monotone with α and uniformly Lipschitz continuous with M. If T_1^k is continuously differentiable and z^* is the zero point of T, then when $\{\nabla T_1^k(z^*)\}$ converges and we take H_k with $H_k - \nabla T_1^k(z^*) \to 0$, then there must exist step sizes $\{\lambda_k\}$ with $\lambda_k \to 1$ such that the $\{z^k\}$ from (1.7) superlinearly converges to z^* for any initial z^0.*

These two theorems are of theoretical significance because the best convergence rate was proved before these theorems is Q-linear for the FBS method. It is also possible to apply them to nonlinear optimization, variational inequalities, etc.

Chapter 6: The influence of step sizes and norms is investigated numerically and theoretically and some strategies for the choices of step sizes, norms and splitting of the operator in the EFBS scheme are discussed in Chapter 6. Those are important because the different choices of step sizes, norms and splitting strategies might cause the speed of the convergence very different. Some examples of practical estimates of Lipschitz constants, strongly monotone moduli for an operator are given in Section 4. One very important part in this dissertation has been given in this chapter in which we considered the influences of the step sizes λ_k, splitting choices of $T = T_1 + T_2$, choices of the operators H_k, on the convergence of the extended FBS algorithms in general. From the knowledge and experience of author, no one has previously considered similar problems for operator splitting methods. The problems and results are new. One of the key results is:

Theorem 6 (Choices of $T = T_1 + T_2$ and H_k). *One of the global optimal choices of the splitting $T = T_1^k + T_2^k$ and the operators H_k in (1.6) for extended linear-quadratic programming problems are*

$$T_1^k \equiv \begin{pmatrix} & -D^T \\ D & \end{pmatrix}, \qquad H_k \equiv \begin{pmatrix} P & \\ & Q \end{pmatrix} \tag{1.8}$$

from the conclusions in Section 6.5 and Section 6.6 (Theorem 6.5.1, Theorem 6.5.2, Theorem 6.6.4, and Theorem 6.6.7), where

$$T(u,v) = \begin{pmatrix} P & -D^T \\ D & Q \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \mathcal{N}_U(u) + p \\ \mathcal{N}_V(v) - q \end{pmatrix}. \tag{1.9}$$

The problems for these choices are generally formulated in Section 6.5 and Section 6.6, and they are hard to solve for general operators T and H_k. They have very important practical significance. More investigation may be required for more general operators with some numerical experiments.

Another important result in this chapter is

Theorem 7 (Choices of λ_k). *Assume that $H_k = S_k^2$ is strongly monotone for every $k \geq 0$ in (1.7). If $\bar{T}_1^k = S_k^{-1} T_1^k S_k^{-1}$ and $\bar{T}_2^k = S_k^{-1} T_2^k S_k^{-1}$, then the global optimal step sizes $\bar{\lambda}_k$ and corresponding linear convergence rate $\bar{c}_k$ of EFBS methods from (1.7) are as follows:*

(a) *Suppose Condition A is uniformly satisfied by $\{\bar{T}_1^k\}$ and $\{\bar{T}_2^k\}$. If M_k and α_k are the Lipschitz constant of $\bar{T}_1^k$ and modulus of $\bar{T}_2^k$, then*

$$\bar{\lambda}_k = \frac{\alpha_k}{M_k^2}, \qquad \bar{c}_k = \frac{M_k^2}{\alpha_k^2 + M_k^2}.$$

(b) *Suppose Condition B is uniformly satisfied by $\{\bar{T}_1^k\}$ and $\{\bar{T}_2^k\}$. If M_k, α'_k and α''_k are the Lipschitz constant of $\bar{T}_1^k$, the moduli of $\bar{T}_1^k$ and $\bar{T}_2^k$, respectively, then*

$$\bar{\lambda}_k = \frac{\alpha'_k + \alpha''_k}{M_k^2 + \alpha'_k \alpha''_k}, \qquad \bar{c}_k = \frac{M_k^2 - (\alpha'_k)^2}{M_k^2 + 2\alpha'_k \alpha''_k + (\alpha''_k)^2}.$$

PART (III): Application

Chapter 7: The application of the EFBS methods to extended linear quadratic programming is discussed in Chapter 7. Simplifications of the continuous time models and the multistage models in the extended linear quadratic programming are analyzed. Two EFBS algorithms for general convex-concave saddle point problems are suggested and their Q-linear convergence are proved under mild assumption. The relationship of the algorithm convergence and the primal-dual gaps of the problem is revealed. Issues of primal-dual gaps, Lagrangian functions and line searches are also considered in this chapter. Details of the algorithm implementation for the multistage models and some numerical experiments are given at last. The numerical results confirm our theoretic conclusions of the previous chapters.

Chapter 8: In Chapter 8, the EFBS method is applied to optimal control of discrete-time systems. The original problem is transformed into a saddle point problem which follows our algorithm structure. Issues of penalty operator (regularization) techniques, ways to strong monotonicity of operators, line searches, splitting of $T = T_1 + T_2$, and choices of H_k are discussed. Two numerical algorithms are presented. At last, numerical experiments for several problems from published literature are given. The EFBS method shows very encouraging numerical results compared with the interior point method by Wright (1992) (we have solved problems involving 1.4 million variables).

Chapter 9: Some applications of the EFBS method to variational inequalities, stochastic programming, and chemical engineering are presented in Chapter 9. The potential applications are discussed in Section 9.5. The relationship with some existing algorithms are given in Section 9.4. Some numerical experiments for the MPC control design in chemical engineering are implemented and the encouraging results are presented in Section 9.3.

PART (IV): Reference

In the last part, a group of reference papers and books will be listed.