# Chapter 3

# THE SPECIAL SPLITTING STRUCTURE IN EXTENDED LINEAR-QUADRATIC MODELS

This chapter describes the multistage and continuous-time extended linear-quadratic programming problems and their basic properties. From the analysis of their special splitting structure, which is the motivation for this dissertation, we find that the forward-backward splitting method seems the most suitable for the numerical structures of demonstrated problems.

## 3.1. Multistage Models in Extended Linear-Quadratic Programming

Many large-scale problems in dynamic and stochastic optimization and optimal control can be modeled with the optimization problems of extended linear-quadratic programming. In general the objective functions in such problems are only piecewise smooth and must be minimized or maximized relative to polyhedral sets of high dimensionality. These large-scale problems have special structure, especially for the multistage or continuous-time case.

As we have seen in Section 2.2, extended linear-quadratic programming, from the Lagrangian representation, is to find a saddle point of a Lagrangian function

$$L(u,v) = p{\cdot}u + \tfrac{1}{2}u{\cdot}Pu + q{\cdot}v - \tfrac{1}{2}v{\cdot}Qv - v{\cdot}Du \quad \text{on} \quad U \times V, \tag{3.1.1}$$

where $U$, $V$, and $P$, $Q$, $D$ have the same definition as those in Section 2.2.

The objective functions of the primal problem (2.2.11) and the dual problem (2.2.12) of (3.1.1) have the following property:

**Theorem 3.1.1 (Rockafellar 1987).** *The objective functions $f$ in (2.2.11) and $g$ in (2.2.12) in Section 2.2 are piecewise linear-quadratic: in each case the space can be partitioned in principle into a finite collection of polyhedral cells, relative to which the function has a linear or quadratic formula. Moreover, $f$ is convex* *while $g$ is concave. When $P$ and $Q$ are positive definite, $f$ is strongly convex and $g$ is strongly concave, both functions have continuous first derivatives.*

Now let us consider a special case of multistage models with dynamical structure (ref. Rockafellar 1991). In problems involving a linear process operating in discrete time, the dynamics can generally be expressed by a system

$$x_t = A_t x_{t-1} + B_t u_t + b_t \quad \text{for} \quad t = 1, \ldots, N, \quad \text{with} \quad x_0 = B_0 u_0 + b_0, \tag{3.1.2}$$

where $u_t$ is a so-called *control* vector in $\mathbb{R}^{k_t}$, and $x_t$ is a *state* vector in $\mathbb{R}^{n_t}$. The control vectors give the true decision variables, while the state vectors may just be artificial constructs, but useful nevertheless. From (3.1.2), the state vectors and control vectors satisfy certain affine expressions as follows:

$$\bar{G}x = Bu + b \quad \text{or} \quad x = \bar{G}^{-1}Bu + \bar{G}^{-1}b, \tag{3.1.3}$$

where

$$\bar{G} = \begin{pmatrix} I & & & \\ -A_1 & I & & \\ & \ddots & \ddots & \\ & & -A_N & I \end{pmatrix}, \quad B = \begin{pmatrix} B_0 & & & \\ & B_1 & & \\ & & \ddots & \\ & & & B_N \end{pmatrix}, \tag{3.1.4}$$

$$u = \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_N \end{pmatrix}, \qquad x = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{pmatrix}, \qquad b = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_N \end{pmatrix}.$$

Hence $x$ could therefore be eliminated in principle from the formulation of any optimization problem with respect to $u$ from (3.1.3). For a number of reasons, however, it seems best to keep them. Since the dimension $k_t$ of $u_t$ and the dimension $n_t$ of $x_t$ are allowed to vary with $t$, there is no loss of generality in taking (3.1.2) as the form for the dynamics. Even if a problem may not seem to involve state vectors, they can always be introduced so as to develop this type of expression (see Rockafellar and Wets, 1990, for details).

The model problem here to consider for such a dynamical system is

$$
\begin{aligned}
&\text{minimize subject to (3.1.2) with } u_t \in U_t \text{ for } t = 0, 1 \ldots, N:\\
&\quad f(u_0, \ldots, u_N) := p_0 \cdot u_0 + \tfrac{1}{2} u_0 \cdot P_0 u_0\\
(\mathcal{P}_d) \qquad &\qquad + \sum_{t=1}^{N} \big[ p_t \cdot u_t + \tfrac{1}{2} u_t \cdot P_t u_t + \rho_{V_t, Q_t}(q_t - C_t x_{t-1} - D_t u_t)\\
&\qquad - c_t \cdot x_{t-1} \big] + \rho_{V_{N+1}, Q_{N+1}}(q_{N+1} - C_{N+1} x_N) - c_{N+1} \cdot x_N,
\end{aligned}
$$

where the sets $U_t \subset \mathbb{R}^{k_t}$ and $V_t \subset \mathbb{R}^{l_t}$ are polyhedral and the matrices $P_t \in \mathbb{R}^{k_t \times k_t}$ and $Q_t \in \mathbb{R}^{l_t \times l_t}$ are positive semidefinite. (The elements $V_t$, $Q_t$, $q_t$, $C_t$ and $c_t$ are indexed from $t = 1$ to $t = N + 1$ instead of from $t = 0$ to $t = N$ for the sake of achieving a symmetric formulation of duality.)

Problem $(\mathcal{P}_d)$ was introduced in Rockafellar and Wets (1990). It was shown in that paper that $(\mathcal{P}_d)$ is the primal problem associated with the Lagrangian $L(u, v)$ on $U \times V$, here

$$
\begin{aligned}
&U := U_0 \times \cdots \times U_N,\\
&V := V_1 \times \cdots \times V_{N+1},\\
&L(u, v) := [p_0 \cdot u_0 + \tfrac{1}{2} u_0 \cdot P_0 u_0] + [q_{N+1} \cdot v_{N+1} - \tfrac{1}{2} v_{N+1} \cdot Q_{N+1} v_{N+1}]\\
&\qquad + \sum_{t=1}^{N} \big[ p_t \cdot u_t + \tfrac{1}{2} u_t \cdot P_t u_t + q_t \cdot v_t - \tfrac{1}{2} v_t \cdot Q_t v_t - v_t \cdot D_t u_t \big] - l(u, v),
\end{aligned}
\tag{3.1.5}
$$

$v$ is defined in (3.1.9) and the expression $l(u, v)$ is affine separately in $u$ and $v$ and is defined from the dynamical equations (3.1.2) and a dual dynamical system by

$$
l(u, v) = \sum_{t=1}^{N+1} x_{t-1} \cdot (C_t^* v_t + c_t) = \sum_{t=0}^{N} y_{t+1} \cdot (B_t u_t + b_t). \tag{3.1.6}
$$

here the dual state vectors $y_t \in \mathbb{R}^{n_t}$ are determined by the following dual dynamical system which goes backward in time with the vectors $v_t \in \mathbb{R}^{l_t}$ as controls:

$$
\begin{aligned}
y_t &= A_t^* y_{t+1} + C_t^* v_t + c_t \quad \text{for} \quad t = N, \ldots, 1,\\
&\text{with} \quad y_{N+1} = C_{N+1}^* v_{N+1} + c_{N+1}.
\end{aligned}
\tag{3.1.7}
$$

Similarly, from (3.1.7), the dual state vectors and control vectors satisfy an affine expression as follows:

$$\bar{G}^* y = C^* v + c \quad \text{or} \quad x = (\bar{G}^*)^{-1} C v + (\bar{G}^*)^{-1} c, \tag{3.1.8}$$

where $\bar{G}^*$ is the transpose of $\bar{G}$ and $C = \text{diag}\{C_1, \cdots, C_{N+1}\}$,

$$v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ u_{N+1} \end{pmatrix}, \qquad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{N+1} \end{pmatrix}, \qquad c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{N+1} \end{pmatrix}, \tag{3.1.9}$$

**Theorem 3.1.2 (Rochafellar and Wets 1990).** *A control sequence $\bar{u} = (\bar{u}_0, \ldots, \bar{u}_N)$ solves the multistage optimization problem $(\mathcal{P}_d)$ if and only if there is a dual control sequence $\bar{v} = (\bar{v}_1, \ldots, \bar{v}_{N+1})$ such that $(\bar{u}, \bar{v})$ is a saddle point of $L(u, v)$ relative to $U \times V$ in (3.1.5).*

From (3.1.5), (3.1.6) and (3.1.7) the dual problem is derived as

$$(\mathcal{Q}_d) \qquad \begin{aligned} &\text{maximize subject to (3.1.7) with } v_t \in V_t \text{ for } t = 1 \ldots, N+1 : \\ &g(v_1, \ldots, v_{N+1}) := q_{N+1} \cdot v_{N+1} - \tfrac{1}{2} v_{N+1} \cdot Q_{N+1} v_{N+1} \\ &\quad + \sum_{t=1}^{N} \big[ q_t \cdot v_t - \tfrac{1}{2} v_t \cdot Q_t v_t - \rho_{U_t, P_t}(B_t^* y_{t+1} + D_t^* v_t - p_t) \\ &\quad - b_t \cdot y_{t+1} \big] - \rho_{U_0, P_0}(B_0^* y_1 - p_0) - b_0 \cdot y_1 . \end{aligned}$$

Since $L(u, v)$ in (3.1.5) is, at most, quadratic in $u$ and $v$, while the sets $U$ and $V$ are polyhedral, $(\mathcal{P}_d)$ is in fact a problem of extended linear-quadratic programming in standard form, and so is its dual $(\mathcal{Q}_d)$, except for a change of sign in converting to maximization.

**Theorem 3.1.3 (Rockafellar and Wets 1990).** *Problem $(\mathcal{P}_d)$ has a solution if and only if its objective $f(u)$ is bounded below on $U$ and is finite for at least one $u \in U$. Then* $\min(\mathcal{P}_d) = \max(\mathcal{Q}_d)$.

Rockafellar and Wets also proved that the saddle point problem associated with $L$ on $U \times V$ has some decomposability properties. However, most of the difficulties for solving it are caused by $l(u,v)$ and the dynamical systems (3.1.2) and (3.1.7) in practice.

## 3.2. Continuous-Time Models in Extended Linear-Quadratic Programming

Now we shall review the continuous-time models by Rockafellar (1987). First of all, we recall the problem as follows:

The time interval $[t_0, t_1]$ is fixed in the following.

The model problem we propose to consider is:

over the dynamical system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + b(t) \quad \text{a.e.,} \quad x(t_0) = B_e u_e + b_e \tag{3.2.1}$$

with control space

$$\mathcal{U} = \{(u, u_e) \,\big|\, u \in \mathcal{L}^2, u(t) \in U(t) \quad \text{a.e.,} \quad u_e \in U_e\} \tag{3.2.2}$$

minimize the functional

$$(\mathcal{P}_c) \qquad \begin{aligned} \mathcal{F}(u, u_e) := &\int_{t_0}^{t_1} [p(t){\cdot}u(t) + \tfrac{1}{2}u(t){\cdot}P(t)u(t) - c(t){\cdot}x(t)]\, dt \\ &+ [p_e{\cdot}u_e + \tfrac{1}{2}u_e{\cdot}P_e u_e - c_e{\cdot}x(t_1)] + \rho_{V_e,Q_e}(q_e - C_e x(t_1) \\ &- D_e u_e) + \int_{t_0}^{t_1} \rho_{V(t),Q(t)}(q(t) - C(t)x(t) - D(t)u(t))\, dt. \end{aligned}$$

where the sets $U(t) \subset \mathbb{R}^k$, $U_e \subset \mathbb{R}^{k_e}$, $V(T) \subset \mathbb{R}^l$ and $V_e \subset \mathbb{R}^{l_e}$ are polyhedral convex and the matrices $P(t) \in \mathbb{R}^{k\times k}$, $P_e \in \mathbb{R}^{k_e\times k_e}$, $Q(t) \in \mathbb{R}^{l\times l}$ and $Q_e \in \mathbb{R}^{l_e\times l_e}$ are positive semidefinite. All the data elements in problem $(\mathcal{P}_c)$ are assumed to depend continuously on t. The $\rho$ terms are defined by (2.2.2):

$$\rho_{V,Q}(s) = \sup_{v\in V}\{s{\cdot}v - \tfrac{1}{2}v{\cdot}Qv\}. \tag{3.2.3}$$

Problem $(\mathcal{P}_c)$ was shown in [273] that it is the primal problem associated with the Lagrangian $\mathcal{J}(u, u_e; v, v_e)$ on $\mathcal{U} \times \mathcal{V}$, where

$$\begin{aligned}
\mathcal{V} &:= \{(v, v_e) \,\big|\, v \in \mathcal{L}^2, v(t) \in V(t) \quad \text{a.e.}, \quad v_e \in V_e\}, \\
\mathcal{J}(u, u_e; v, v_e) &:= \int_{t_0}^{t_1} J(t, u(t), v(t))\, dt \\
&\qquad + J_e(u_e, v_e) - [(u, u_e), (v, v_e)]
\end{aligned} \tag{3.2.4}$$

and here

$$\begin{aligned}
J(t, u, v) &:= p(t)\cdot u + q(t)\cdot v + \tfrac{1}{2} u\cdot P(t) u - \tfrac{1}{2} v\cdot Q(t) v - v\cdot D(t) u, \\
J_e(u_e, v_e) &:= p_e\cdot u_e + q_e\cdot v_e + \tfrac{1}{2} u_e\cdot P_e u_e - \tfrac{1}{2} v_e\cdot Q_e v_e - v_e\cdot D_e u_e, \\
[(u, u_e), (v, v_e)] &:= \int_{t_0}^{t_1} x(t)\cdot[C^*(t) v(t) + c(t)]\, dt + x(t_1)\cdot[{C_e}^* v_e + c_e] \\
&= \int_{t_0}^{t_1} y(t)\cdot[B(t) u(t) + b(t)]\, dt + y(t_0)\cdot[B_e u_e + b_e]
\end{aligned} \tag{3.2.5}$$

Thus we called $x(t)$ and $(u(t), u_e)$ as *primal* state and control vectors while $y(t)$ and $(v(t), v_e)$ are called as *dual* state and control vectors. Here $y(t)$ satisfies the dynamical system

$$-\dot{y}(t) = A^*(t) y(t) + C^*(t) v(t) + c(t) \quad \text{a.e.}, \quad y(t_1) = C_e^* v_e + c_e. \tag{3.2.6}$$

The dual problem of $(\mathcal{P}_c)$ is:

over the dual dynamical system (3.2.6) with dual control space $\mathcal{V}$ defined in (3.2.4),

maximize the functional

$$(\mathcal{Q}_c) \qquad \begin{aligned}
\mathcal{G}(v, v_e) &:= \int_{t_0}^{t_1} [q(t)\cdot v(t) - \tfrac{1}{2} v(t)\cdot Q(t) v(t) - b(t)\cdot y(t)]\, dt \\
&\quad + [q_e\cdot v_e - \tfrac{1}{2} v_e\cdot Q_e v_e - b_e\cdot y(t_0)] + \rho_{U_e, P_e}(B_e^* y(t_0) - p_e \\
&\quad + D_e^* v_e) + \int_{t_0}^{t_1} \rho_{U(t), P(t)}(B^*(t) y(t) + D^*(t) v(t) - p(t))\, dt.
\end{aligned}$$

Let us denote by $\{(\mathcal{P}^1), (\mathcal{Q}^1)\}$ and $\{(\mathcal{P}^\infty), (\mathcal{Q}^\infty)\}$ the modified forms of the problem pair $\{(\mathcal{P}_c), (\mathcal{Q}_c)\}$ in which the control functions $u : t \mapsto u(t)$ and

$v : t \mapsto v(t)$ are required to be $\mathcal{L}^1$ and $\mathcal{L}^\infty$, respectively, rather than $\mathcal{L}^2$. The optimal values in these problems satisfy

$$\inf(\mathcal{P}^\infty) \geq \inf(\mathcal{P}_c) \geq \inf(\mathcal{P}^1) \geq \sup(\mathcal{Q}^1) \geq \sup(\mathcal{Q}_c) \geq \sup(\mathcal{Q}^\infty).$$

in general.

**Theorem 3.2.1 ([273], Thm. 6.3).** *Under the assumption that the primal and dual finiteness conditions both hold, one has*

$$-\infty < \min(\mathcal{P}^1) = \max(\mathcal{Q}^1) < \infty.$$

*In other words, optimal controls exist for both problems and the corresponding objective values are finite and equal.*

**Theorem 3.2.2 ([275], Thm. 2.3).** *In the case $\min(\mathcal{P}^1) = \max(\mathcal{Q}^1)$, the following condition is both necessary and sufficient in order that $(\bar{u}, \bar{u}_e)$ be optimal in $(\mathcal{P}^1)$ and $(\bar{v}, \bar{v}_e)$ be optimal in $(\mathcal{Q}^1)$. The condition is: $(\bar{u}, \bar{u}_e; \bar{v}, \bar{v}_e)$ is a saddle point of $\mathcal{J}(u, u_e; v, v_e)$ relative to $(u, u_e) \in \mathcal{U}$ with $u \in \mathcal{L}^1$ and $(v, v_e) \in \mathcal{V}$ with $v \in \mathcal{L}^1$.*

**Theorem 3.2.1 ([275], Thm. 3.1).** *Under the assumption that the primal and dual finiteness conditions both hold, every optimal solution $(\bar{u}, \bar{u}_e)$ to $(\mathcal{P}^1)$ actually has $\bar{u} \in \mathcal{L}^\infty$ and therefore is an optimal solution to $(\mathcal{P}^\infty)$; likewise, every optimal solution $(\bar{v}, \bar{v}_e)$ to $(\mathcal{Q}^1)$ has $\bar{v} \in \mathcal{L}^\infty$ and therefore is an optimal solution to $(\mathcal{Q}^\infty)$. Thus,*

$$-\infty < \min(\mathcal{P}^\infty) = \max(\mathcal{Q}^\infty) < \infty.$$

Therefore, combining the above, one has that solving the saddle point problem associated with $\mathcal{J}(u, u_e; v, v_e)$ on $\mathcal{U} \times \mathcal{V}$ is equivalent to solving its primal problem $(\mathcal{P}_c)$ and its dual problem $(\mathcal{Q}_c)$ when the primal and dual finiteness conditions both hold.

Note: the corresponding Hilbert space $\mathcal{H}$ is given in (2.3.17).

Now let us focus our attention on the ODEs (ordinary differential equations) (3.2.1) and (3.2.6). We shall use $\dot{z}$ to express $\frac{d}{dt}z$ later on.

First of all, consider the differential equations:

$$\dot{x}(t) = A(t)x(t), \quad x(t_0) \text{ given} \tag{3.2.7}$$

$$-\dot{y}(t) = A^*(t)y(t), \quad y(t_1) \text{ given} \tag{3.2.8}$$

where $t_1 > t_0$.

From the ODE theory, we know that if the matrix $A(t)$ has elements which are piecewise continuous functions of time $[t_0, \infty)$, then the ODE (3.2.7) has an unique solution defined on $[t_0, \infty)$ and the solution is given by

$$x(t) = \Phi(t, t_0)x(t_0), \quad \text{with} \quad \dot{\Phi}(t, \tau) = A(t)\Phi(t, \tau), \Phi(\tau, \tau) = I \tag{3.2.9}$$

Where $\Phi(t, \tau)$ is called transition matrix. It has the following properties:
(a) $\Phi(t, \tau)$ is invertible.
(b) $\Phi(t, \tau) = \Phi(t, r)\Phi(r, \tau)$ for any $r \in (t, \tau)$.
(c) $\Phi(t, \tau) = \Phi^{-1}(\tau, t)$.
(d) for constant matrix $A \neq A(t)$, $\Phi(t, \tau) = \exp[A{\cdot}(t - \tau)]$ with

$$\exp[A{\cdot}(t - \tau)] = \sum_{k=0}^{\infty} \frac{1}{k!}(t - \tau)^k A^k$$

**Lemma 3.2.3.** *If the solution of (3.2.7) is given by (3.2.9), then the solution of ODE (3.2.8) is as follows.*

$$y(t) = \Phi^*(t_1, t)y(t_1) \tag{3.2.10}$$

**Proof.**

Since $\; -\dot{y}(t) = A^*(t)y(t)$ and $\dot{\Phi}(t, \tau) = A(t)\Phi(t, \tau)$,

thus, $\; -\Phi^*(t, \tau)\dot{y}(t) = \Phi^*(t, \tau)A^*(t)y(t) = \dot{\Phi}^*(t, \tau)y(t)$.

This implies $\dfrac{d}{dt}[\Phi^*(t, \tau)y(t)] = 0$.

Hence $\Phi^*(t, \tau)y(t) \equiv \Phi^*(\tau, \tau)y(\tau) = y(\tau)$.

Let $\tau = t_1$. Then $\Phi^*(t, t_1)y(t) = y(t_1)$.

Therefore, $y(t) = [\Phi^*(t, t_1)]^{-1}y(t_1) = \Phi^*(t_1, t)y(t_1)$

□

Now let us consider the nonhomogeneous problems.

$$\dot{x}(t) = A(t)x(t) + b(t), \qquad x(t_0) \ \text{ given.} \tag{3.2.11}$$

$$-\dot{y}(t) = A^*(t)y(t) + c(t), \qquad y(t_1) \ \text{ given.} \tag{3.2.12}$$

**Lemma 3.2.4.** *For $\Phi(\tau,t) = \Phi^{-1}(t,\tau)$, we have*

$$\begin{aligned} \dot{\Phi}(\tau,t) &= -\Phi(\tau,t)A(t), \\ \dot{\Phi}^*(\tau,t) &= -A^*(t)\Phi^*(\tau,t). \end{aligned} \tag{3.2.13}$$

**Proof.** We only need to prove the first equality.

$$\begin{aligned} &\text{Since} \quad \Phi(t,\tau)\Phi(\tau,t) = \Phi(\tau,t)\Phi(t,\tau) = I, \\ &\text{hence} \quad \dot{\Phi}(t,\tau)\Phi(\tau,t) + \Phi(t,\tau)\dot{\Phi}(\tau,t) = 0. \\ &\text{But} \quad \dot{\Phi}(t,\tau) = A(t)\Phi(t,\tau). \\ &\text{So} \quad A(t) + \Phi(t,\tau)\dot{\Phi}(\tau,t) = 0. \\ &\text{Or} \quad \dot{\Phi}(\tau,t) = -\Phi^{-1}(t,\tau)A(t) = -\Phi(\tau,t)A(t). \end{aligned}$$

where 0 and $I$ in the above means the zero matrix and the identity matrix. □

**Proposition 3.2.5.** *If the solutions of (3.2.7) and (3.2.8) are given by (3.2.9) and (3.2.10), then the solutions of (3.2.11) and (3.2.12) are given as follows:*

$$\begin{aligned} x(t) &= \Phi(t,t_0)x(t_0) + \int_{t_0}^{t} \Phi(t,\tau)b(\tau)\,d\tau, \\ y(t) &= \Phi^*(t_1,t)y(t_1) + \int_{t}^{t_1} \Phi^*(\tau,t)c(\tau)\,d\tau. \end{aligned} \tag{3.2.14}$$

**Proof.** The initial and terminal conditions are obviously satisfied. Now from (3.2.14) and Lemma 3.2.4, we have

$$\begin{aligned}
\dot{x}(t) &= \dot{\Phi}(t,t_0)x(t_0) + \Phi(t,t)b(t) + \int_{t_0}^{t} \dot{\Phi}(t,\tau)b(\tau)\,d\tau \\
&= A(t)[\Phi(t,t_0)x(t_0) + \int_{t_0}^{t} \Phi(t,\tau)b(\tau)\,d\tau] + b(t) \\
&= A(t)x(t) + b(t). \\
\dot{y}(t) &= \dot{\Phi}^*(t_1,t)y(t_1) - \Phi^*(t,t)c(t) + \int_{t}^{t_1} \dot{\Phi}^*(\tau,t)c(\tau)\,d\tau \\
&= -A^*(t)\Phi^*(t_1,t)y(t_1) - c(t) - A^*(t)\int_{t}^{t_1} \Phi^*(\tau,t)c(\tau)\,d\tau \\
&= -A^*(t)y(t) - c(t).
\end{aligned}$$

Therefore $x(t)$ and $y(t)$ in (3.2.14) are the solutions of (3.2.11) and (3.2.12). □

**Corollary 3.2.6.** *If the solutions of (3.2.7) and (3.2.8) are given by (3.2.9) and (3.2.10), then the solutions of (3.2.1) and (3.2.6) are given as follows:*

$$\begin{aligned}
x(t) &= \Phi(t,t_0)(B_e u_e + b_e) + \int_{t_0}^{t} \Phi(t,\tau)\big[B(\tau)u(\tau) + b(\tau)\big]\,d\tau, \\
y(t) &= \Phi^*(t_1,t)(C_e^* v_e + c_e) + \int_{t}^{t_1} \Phi^*(\tau,t)\big[C^*(\tau)v(\tau) + c(\tau)\big]\,d\tau.
\end{aligned} \tag{3.2.15}$$

**Proof.** (3.2.15) is a direct conclusion of Proposition 3.2.5 by replacing corresponding terms. □

## 3.3. Some Conclusions about Linear-Quadratic Regulators

In the next section, we shall deel with the problems of linear-quadratic regulators when the splitting strategies are discussed. Hence, a discussion in order on some properties of linear-quadratic regulators, which are special cases of the models in Section 3.1 and Section 3.2, in this section.

*3.3.1. Continuous-time linear-quadratic regulators.*

Consider the following regulator problems.

$$(\mathcal{RP}) \qquad \min_u \int_{t_0}^{t_1} [\tfrac{1}{2}u\cdot P(t)u + \tfrac{1}{2}x\cdot R(t)x]\,dt + \tfrac{1}{2}x(t_1)\cdot R_1 x(t_1)$$
$$\text{with} \quad \dot{x}(t) = A(t)x(t) + B(t)u(t), \qquad x(t_0) = b_0$$

and

$$(\mathcal{R}_1) \qquad \min_u \int_{t_0}^{t_1} [\tfrac{1}{2}u\cdot P(t)u + \tfrac{1}{2}x\cdot R(t)x - c(t)\cdot x]\,dt + \tfrac{1}{2}x(t_1)\cdot R_1 x(t_1) + c_1\cdot x(t_1)$$

$$(\mathcal{R}_2) \qquad \min_u \int_{t_0}^{t_1} [\tfrac{1}{2}u\cdot P(t)u + p(t)\cdot u + \tfrac{1}{2}x\cdot R(t)x - c(t)\cdot x]\,dt + \tfrac{1}{2}x(t_1)\cdot R_1 x(t_1) + c_1\cdot x(t_1)$$

with the same dynamical system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + b(t), \qquad x(t_0) = b_0 \tag{3.3.1}$$

as well as

$$(\mathcal{R}_3) \qquad \min_w \int_{t_0}^{t_1} [\tfrac{1}{2}w\cdot G(t)w + \tfrac{1}{2}x\cdot F(t)x + x\cdot N(t)w + p'(t)\cdot w - c'(t)\cdot x]\,dt$$
$$\text{with} \quad \dot{x}(t) = A'(t)x(t) + B'(t)w(t) + b(t), \quad x(t_0) = b_0$$

where $P(t)$ and $G(t)$ are symmetric positive-definite for $t \in [t_0, t_1]$ and $R(t)$, $F(t)$, $R_1$ are positive semidefinite for $t \in [t_0, t_1]$.

From known theory, $(\mathcal{R}_3)$ is just a special case of $(\mathcal{R}_2)$, with $P(t) = G(t)$, $B(t) = B'(t)$, $R(t) = F(t) - N(t)G^{-1}(t)N^*(t)$, $A(t) = A'(t) - B'(t)G^{-1}(t)N^*(t)$, $p(t) = p'(t)$, $u(t) = w(t) + G^{-1}(t)N^*(t)x(t)$ and $c(t) = c'(t) + N(t)G^{-1}(t)p'(t)$. Thus it is not necessary to discuss it here.

As for the relations among $(\mathcal{RP})$, $(\mathcal{R}_1)$, and $(\mathcal{R}_2)$ we have the following theorem.

**Theorem 3.3.1.** *If the matrix Riccati equation of* $(\mathcal{RP})$

$$\dot{S} = SB(t)P^{-1}(t)B^*(t)S - SA(t) - A^*(t)S - R(t), \quad S(t_1) = R_1 \tag{3.3.2}$$

*has a solution* $S(t)$ *and let* $K(t) = P^{-1}(t)B^*(t)S(t)$, *then the optimal control laws of* $(\mathcal{RP})$, $(\mathcal{R}_1)$ *and* $(\mathcal{R}_2)$ *are respectively*

$$u(t) = -K(t)x(t), \tag{3.3.3}$$

$$u(t) = -K(t)x(t) - P^{-1}(t)B^*(t)g_1(t), \tag{3.3.4}$$

$$u(t) = -K(t)x(t) - P^{-1}(t)B^*(t)g_2(t) - P^{-1}(t)p(t), \tag{3.3.5}$$

*where* $g_i(t)$ *for* $i = 1, 2$ *satisfy the first order linear ODEs:*

$$\begin{aligned} \dot{g}_i(t) =& [S(t)B(t)P^{-1}(t)B^*(t) - A^*(t)]g_i(t) \\ &- S(t)h_i(t) + c(t), \qquad g_i(t_1) = c_1 \end{aligned} \tag{3.3.6}$$

*and* $h_1(t) = b(t)$, $h_2(t) = b(t) - B(t)P^{-1}(t)p(t)$.

**Proof.** We only need to prove (3.3.5) since $(\mathcal{R}_1)$ and $(\mathcal{RP})$ are the special cases of $(\mathcal{R}_2)$.

The Hamiltonian of $(\mathcal{R}_2)$ is

$$H = \tfrac{1}{2}u{\cdot}P(t)u + \tfrac{1}{2}x{\cdot}R(t)x + p(t){\cdot}u - c(t){\cdot}x + \sigma(t){\cdot}[A(t)x + B(t)u + b(t)]$$

From the optimality conditions about $H$, one can obtain the control $u$ satisfying

$$u(t) = -P^{-1}(t)B^*(t)\sigma(t) - P^{-1}(t)p(t) \tag{3.3.7}$$

and the Euler-Lagrange equations as follows:

$$\begin{aligned} \dot{x} &= A(t)x - B(t)P^{-1}(t)B^*(t)\sigma + h_2(t), \quad x(t_0) = b_0, \\ \dot{\sigma} &= -R(t)x - A^*(t)\sigma + c(t), \quad \sigma(t_1) = R_1x(t_1) + c_1. \end{aligned} \tag{3.3.8}$$

where $h_2(t) = b(t) - B(t)P^{-1}(t)p(t)$.

Solving the E-L equations (3.3.8) by the sweep method. i.e., let

$$\sigma(t) = S(t)x(t) + g_2(t) \tag{3.3.9}$$

Replacing (3.3.9) into the second equation of (3.3.8) gives

$$S\dot{x} + \dot{S}x + \dot{g}_2 = -R(t)x - A^*(t)Sx - A^*(t)g_2 + c(t).$$

From (3.3.9) the first equation of (3.3.8), the above equation becomes

$$\begin{aligned}0 \equiv &[\dot{S}(t) + S(t)A(t) - S(t)B(t)P^{-1}(t)B^*(t)S(t) + R(t) + A^*(t)S(t)]x(t) \\ &+ \dot{g}_2(t) + [A^*(t) - S(t)B(t)P^{-1}(t)B^*(t)]g_2(t) - c(t) + S(t)h_2(t).\end{aligned}$$

Therefore the above gives the matrix Riccati equation (3.3.2) and the linear ODE (3.3.6) for $i = 2$. The terminal conditions come from (3.3.9) and $\sigma(t_1) = R_1 x(t_1) + c_1$.

Thus, if (3.3.2) has a solution $S(t)$, then the optimal control law of $(\mathcal{R}_2)$ is

$$\begin{aligned}u(t) &= -P^{-1}(t)B^*(t)[S(t)x(t) + g_2(t)] - P^{-1}(t)p(t) \\ &= -K(t)x(t) - P^{-1}(t)B^*(t)g_2(t) - P^{-1}(t)p(t)\end{aligned}$$

with $g_2(t)$ satisfying (3.3.6). This is just (3.3.5).

If $p(t) = 0$ in $(\mathcal{R}_2)$, then $(\mathcal{R}_2)$ becomes $(\mathcal{R}_1)$ and hence $g_1(t) = g_2(t)$ in (3.3.5). So we get (3.3.4). Furthermore, if $b(t) = 0$ and $c(t) = 0$ in $(\mathcal{R}_1)$ , then $(\mathcal{R}_1)$ is just $(\mathcal{RP})$ and $h_1(t) = 0$, $c(t) = 0$ in (3.3.6), and hence $g_1(t) = 0$ in (3.3.4). Thus we get (3.3.3). □

From the optimal control theory, the matrix Riccati differential equation (3.3.2) has a solution $S(t)$ defined over the entire interval $[t_0, t_1]$ and $S(t)$ is symmetric for any $t \in [t_0, t_1]$ if $P(t)$, $R(t)$, $A(t)$ and $B(t)$ are continuous over $[t_0, t_1]$. If we rewrite (3.3.6) using $K(t)$, it becomes

$$-\dot{g}_i(t) = [A(t) - B(t)K(t)]^* g_i(t) + S(t)h_i(t) - c(t), \qquad g_i(t_1) = c_1 \tag{3.3.10}$$

Note that the closed loop system dynamics with (3.3.3), (3.3.4) and (3.3.5) are respectively

$$\dot{x}(t) = [A(t) - B(t)K(t)]x(t), \qquad x(t_0) = b_0, \tag{3.3.11}$$

$$\begin{aligned}\dot{x}(t) = &[A(t) - B(t)K(t)]x(t) + b(t) \\ &- B(t)P^{-1}(t)B^*(t)g_1(t), \quad x(t_0) = b_0,\end{aligned} \tag{3.3.12}$$

$$\dot{x}(t) = [A(t) - B(t)K(t)]x(t) + b(t) \\ - B(t)P^{-1}(t)[B^*(t)g_2(t) + p(t)], \quad x(t_0) = b_0. \tag{3.3.13}$$

So (3.3.10) and (3.3.11), or (3.3.12), or (3.3.13) have dual properties.

When $A(t)$, $B(t)$, $P(t)$ and $R(t)$ are independent of $t$, it is possible to (3.3.2) that a steady state solution exists. For solving a steady state solution, (3.3.2) becomes

$$SBP^{-1}B^*S - SA - A^*S - R = 0. \tag{3.3.14}$$

From the above, if we know the solution of (3.3.11), which is given by a transition matrix, then we also know the solutions of (3.3.12), (3.3.13) and (3.3.10) with known $S(t)$ of (3.3.2). Therefore, if we could solve the standard LQR problem $(\mathcal{RP})$, then we could get the solutions of LQR problems $(\mathcal{R}_1)$, $(\mathcal{R}_2)$ and $(\mathcal{R}_3)$. Hence, the key is to solve $(\mathcal{RP})$. More precisely, the key is to solve the matrix Riccati equation and the closed loop system of $(\mathcal{RP})$. However, the matrix Riccati equation (3.3.2) is difficult to solve in general. Even if $B$, $A$, $P$ and $Q$ are time-invariant, (3.3.2) is still not easy to solve. From the proof of Theorem 3.3.1, we might have alternative ways to solve $(\mathcal{RP})$. If we could solve the Euler-Lagrange equations of (3.3.8) with $h_2 \equiv 0$, $c \equiv 0$, and $c_1 = 0$, then the optimal solution of $(\mathcal{RP})$ could be given by $\sigma(t)$ and we would not have to solve the difficult equation (3.3.2) and the closed loop ODE (3.3.11), i.e., if we could solve the following equations:

$$\begin{bmatrix} \dot{x} \\ \dot{\sigma} \end{bmatrix} = \begin{bmatrix} A(t) & -B(t)P^{-1}(t)B^*(t) \\ -R(t) & -A^*(t) \end{bmatrix} \begin{bmatrix} x \\ \sigma \end{bmatrix}, \quad \begin{matrix} x(t_0) = b_0, \\ \sigma(t_1) = R_1 x(t_1), \end{matrix} \tag{3.3.15}$$

then we would have solved $(\mathcal{RP})$. Unfortunately, solving (3.3.15) is still a problematic endeavor. The main difficulties arise from the initial value conditions for $x(t)$ with the unspecified terminal value conditions for $\sigma(t)$, even if they have the time-invariant matrices. The numerical methods for (3.3.15) are hard to get started due to the extraordinary sensitivity of Euler-Lagrange equations (ref. Ch.7 of [ ]). In [ ], the authors suggested two ways to solve E-L equations: (1) by finding the transition matrix between unspecified boundary conditions at one end and specified boundary conditions at the other end; or (2) by "sweeping" the boundary conditions from one end point to the other end point. This involves solving a matrix Riccati equation. However, these two ways are not simpler than solving (3.3.2) and (3.3.10).

Fortunately, there is another way to discuss the LQR problems.

**Theorem 3.3.2.** *Let $\big(x_h(t), \sigma_h(t)\big)$, $\big(x_2(t), \sigma_2(t)\big)$ be the solutions of (3.3.15), (3.3.8) respectively and $\big(x_1(t), \sigma_1(t)\big)$ be the solution of (3.3.8) with $p(t) \equiv 0$. Then the solutions of $(\mathcal{RP})$, $(\mathcal{R}_1)$ and $(\mathcal{R}_2)$ are respectively*

$$u_h(t) = -P^{-1}(t)B^*(t)\sigma_h(t), \tag{3.3.16}$$

$$u_1(t) = -P^{-1}(t)B^*(t)\sigma_1(t), \tag{3.3.17}$$

$$u_2(t) = -P^{-1}(t)B^*(t)\sigma_2(t) - P^{-1}(t)p(t). \tag{3.3.18}$$

**Proof.** They are elementary from the proof of Theorem 3.3.1. ◻

Note: $\big(x_h(t), \sigma_h(t)\big)$ and $\big(x_i(t), \sigma_i(t)\big)$ for $i = 1, 2$ do not have the similar relationship as that of Proposition 3.2.5, due to the initial value conditions for $x(t)$ and the unspecified terminal conditions for $\sigma(t)$.

In next two sections, we shall see that some LQR problems may have unspecified initial value conditions or terminal value conditions. Hence, we consider the following problem:

$$(\mathcal{R}_4) \qquad \min_{(u,u_e)} \int_{t_0}^{t_1} [\tfrac{1}{2}u\cdot P(t)u + p(t)\cdot u + \tfrac{1}{2}x\cdot R(t)x - c(t)\cdot x]\, dt + p_e\cdot u_e$$
$$+ \tfrac{1}{2}u_e\cdot P_e u_e + \tfrac{1}{2}x(t_1)\cdot R_1 x(t_1) + c_1\cdot x(t_1)$$

with the dynamical system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + b(t), \quad \text{a.e.} \qquad x(t_0) = B_e u_e + b_e. \tag{3.3.19}$$

**Theorem 3.3.3.** *The optimality conditions for $(\mathcal{R}_4)$ are*

$$P(t)u(t) + B^*(t)\sigma(t) + p(t) = 0, \qquad P_e u_e + B_e^*\sigma(t_0) + p_e = 0, \tag{3.3.20}$$

$$\dot{\sigma}(t) = -R(t)x(t) - A^*(t)\sigma(t) + c(t), \qquad \sigma(t_1) = R_1 x(t_1) + c_1 \tag{3.3.21}$$

*and (3.3.19).*

**Proof.** Let

$$\begin{aligned} E = \int_{t_0}^{t_1} & \{ \tfrac{1}{2} u \cdot P(t) u + p(t) \cdot u + \sigma(t) \cdot [A(t)x + B(t)u + b(t) - \dot{x}] \\ & + \tfrac{1}{2} x \cdot R(t) x - c(t) \cdot x \} \, dt + \tfrac{1}{2} x(t_1) \cdot R_1 x(t_1) + c_1 \cdot x(t_1) \\ & + \tfrac{1}{2} u_e \cdot P_e u_e + p_e \cdot u_e + \sigma(t_0) \cdot [B_e u_e + b_e - x(t_0)]. \end{aligned} \tag{3.3.22}$$

Then from the traditional way we can obtain the variation of $E$ as

$$\begin{aligned} \delta E = & [\frac{\partial H_e}{\partial x(t_1)} - \sigma(t_1)] \delta x(t_1) + \frac{\partial H_e}{\partial u_e} \delta u_e + \frac{\partial H_e}{\partial \sigma(t_0)} \delta \sigma(t_0) \\ & + \int_{t_0}^{t_1} \big[ (\frac{\partial H}{\partial x} + \dot{\sigma}^*) \delta x + \frac{\partial H}{\partial u} \delta u + (\frac{\partial H}{\partial \sigma} - \dot{x}) \delta \sigma \big] \, dt \end{aligned} \tag{3.3.23}$$

where $H$ is the Hamiltonian of $(\mathcal{R}_4)$:

$$H = \tfrac{1}{2} u \cdot P(t) u + \tfrac{1}{2} x \cdot R(t) x + p(t) \cdot u - c(t) \cdot x + \sigma(t) \cdot [A(t)x + B(t)u + b(t)]$$

which is the same as that of $(\mathcal{R}_2)$ and $H_e$ is the following:

$$\begin{aligned} H_e = & \tfrac{1}{2} x(t_1) \cdot R_1 x(t_1) + c_1 \cdot x(t_1) + \tfrac{1}{2} u_e \cdot P_e u_e \\ & + p_e \cdot u_e + \sigma(t_0) \cdot [B_e u_e + b_e - x(t_0)]. \end{aligned}$$

Hence, if $(u(t), u_e)$ is a solution of $(\mathcal{R}_4)$, then we have from (3.3.23) that

$$\begin{aligned} & \frac{\partial H_e}{\partial x(t_1)} - \sigma(t_1) = 0, \qquad \frac{\partial H_e}{\partial u_e} = 0, \\ & \frac{\partial H}{\partial \sigma} - \dot{x} = 0, \qquad \frac{\partial H}{\partial u} = 0, \\ & \frac{\partial H}{\partial x} + \dot{\sigma}^* = 0, \qquad \frac{\partial H_e}{\partial \sigma(t_0)} = 0. \end{aligned}$$

However, these are just the conditions (3.3.19), (3.3.20) and (3.3.21). □

**Corollary 3.3.4.** *If $P(t)$ and $P_e$ are positive-definite, then solving $(\mathcal{R}_4)$ is equivalent to solving the equations:*

$$\begin{aligned} & \dot{x} = A(t)x - B(t)P^{-1}(t)B^*(t)\sigma + h_2(t), \quad x(t_0) = -\bar{P}_e \sigma(t_0) + \bar{b}_e, \\ & \dot{\sigma} = -R(t)x - A^*(t)\sigma + c(t), \quad \sigma(t_1) = R_1 x(t_1) + c_1 \end{aligned} \tag{3.3.24}$$

*with*

$$
\begin{aligned}
u(t) &= -P^{-1}(t)B^*(t)\sigma(t) - P^{-1}(t)p(t), \\
u_e &= -P_e^{-1}B_e^*\sigma(t_0) - P_e^{-1}p_e,
\end{aligned}
\tag{3.3.25}
$$

*where* $h_2(t) = b(t) - B(t)P^{-1}(t)p(t)$, $\bar{P}_e = B_e P_e^{-1} B_e^*$, *and* $\bar{b}_e = b_e - B_e P_e^{-1} p_e$.

**Proof.** The conclusion is obvious from Theorem 3.3.3. □

**Note:** We can not use the sweep method to solve the Euler-Lagrange equations (3.3.24) of $(\mathcal{R}_4)$. Otherwise, if $\sigma(t) = S(t)x(t) + g(t)$, then $S(t_1)x(t_1) + g(t_1) = R_1 x(t_1) + c_1$, $\sigma(t_0) = S(t_0)x(t_0) + g(t_0)$ and $x(t_0) = -\bar{P}_e\sigma(t_0) + \bar{b}_e$ imply that

$$
S(t_1) = R_1, S(t_0) = -\bar{P}_e^{-1}, g(t_1) = c_1, g(t_0) = \bar{P}_e^{-1}\bar{b}_e.
$$

So we will get a matrix Riccati equation about $S(t)$ with two point boundary value conditions and a first order ODE about $g(t)$ with two point boundary value conditions. These are not correct equations because we only need one initial or terminal value condition for each element of $S(t)$ and $g(t)$. Thus the sweep method can not solve (3.3.24).

*3.3.2. Discrete-time linear-quadratic regulators.*

Now let us turn our attention to discrete-time situation.

Consider the following discrete time linear-quadratic regulator problems

$$
(\mathcal{RP}^d) \qquad \min_u \sum_{t=1}^{N} [\tfrac{1}{2} u_t \cdot P_t u_t + \tfrac{1}{2} x_t \cdot R_t x_t] + \tfrac{1}{2} x_0 \cdot R_0 x_0
$$

$$
\text{with} \quad x_t = A_t x_{t-1} + B_t u_t, \quad \text{for } t = 1, \dots, N, \quad x_0 = b_0
$$

and

$$
(\mathcal{R}_1^d) \qquad \min_u \sum_{t=1}^{N} [\tfrac{1}{2} u_t \cdot P_t u_t + \tfrac{1}{2} x_t \cdot R_t x_t + c_t \cdot x_t] + \tfrac{1}{2} x_0 \cdot R_0 x_0 + c_0 \cdot x_0
$$

$$
(\mathcal{R}_2^d) \qquad \min_u \sum_{t=1}^{N} [\tfrac{1}{2} u_t \cdot P_t u_t + \tfrac{1}{2} x_t \cdot R_t x_t + c_t \cdot x_t + p_t \cdot u_t] + \tfrac{1}{2} x_0 \cdot R_0 x_0 + c_0 \cdot x_0
$$

with the same dynamics

$$x_t = A_t x_{t-1} + B_t u_t + b_t, \quad \text{for } t = 1, \ldots, N, \qquad x_0 = b_0 \tag{3.3.26}$$

Similar to $(\mathcal{R}_3)$, we may have a corresponding discrete-time regulator here. However, it could be transformed into a special case of $(\mathcal{R}_2^d)$ as well. So it is not written out here.

We have a similar result in the continuous time case.

**Theorem 3.3.5.** *If $\{S_0, S_1, \ldots, S_N\}$ is a solution of the discrete matrix Riccati equation of $(\mathcal{RP}^d)$, and let $K_t = P_t^{-1} B_t^* S_t$ $(t = 1, 2, \ldots, N)$, then the optimal control laws of $(\mathcal{RP}^d)$, $(\mathcal{R}_1^d)$ and $(\mathcal{R}_2^d)$ are respectively*

$$\begin{aligned} u_t &= -K_t x_t, && t = 1, 2, \ldots, N; \\ u_t &= -K_t x_t - P_t^{-1} B_t^* g_t, && t = 1, 2, \ldots, N; \\ u_t &= -K_t x_t - P_t^{-1} B_t^* h_t - P_t^{-1} p_t, && t = 1, 2, \ldots, N. \end{aligned} \tag{3.3.27}$$

*Where $g_t$ and $h_t$ satisfy respectively the following for $t = 0, \ldots, N-1$*

$$g_t = A_{t+1}^* [I + K_{t+1}^* B_{t+1}^*]^{-1} (g_{t+1} + S_{t+1} b_{t+1}) + c_t, \quad g_N = c_N, \tag{3.3.28}$$

$$\begin{aligned} h_t =& A_{t+1}^* [I + K_{t+1}^* B_{t+1}^*]^{-1} (h_{t+1} + S_{t+1} b_{t+1} \\ & - K_{t+1}^* p_{t+1}) + c_t, \quad h_N = c_N, \end{aligned} \tag{3.3.29}$$

*and the discrete matrix Riccati equation of $(\mathcal{RP}^d)$ is*

$$\begin{aligned} S_t =& R_t + A_{t+1}^* [S_{t+1}^{-1} + B_{t+1} P_{t+1}^{-1} B_{t+1}^*]^{-1} A_{t+1}, \\ S_N =& R_N, \qquad t = 0, \ldots, N-1. \end{aligned} \tag{3.3.30}$$

**Proof.** Since $(\mathcal{RP}^d)$ and $(\mathcal{R}_1^d)$ are the special cases of $(\mathcal{R}_2^d)$, we only need to prove the third control law in (3.3.27).

Consider the Hamiltonian of $(\mathcal{R}_2^d)$,

$$\begin{aligned} H &= \sum_{t=0}^{N} H_t, \qquad H_0 = \tfrac{1}{2} x_0 \cdot R_0 x_0 + c_0 \cdot x_0 + \sigma_0 \cdot b_0, \\ H_t &= \tfrac{1}{2} u_t \cdot P_t u_t + \tfrac{1}{2} x_t \cdot R_t x_t + c_t \cdot x_t + p_t \cdot u_t + \sigma_t \cdot [A_t x_{t-1} + B_t u_t + b_t]. \end{aligned}$$

Then the optimality conditions give that

$$\begin{aligned} \sigma_{t-1} &= \left(\frac{\partial H}{\partial x_{t-1}}\right)^* = R_{t-1}x_{t-1} + A_t^*\sigma_t + c_{t-1}, \quad \sigma_N = R_N x_N + c_N, \\ 0 &= \frac{\partial H}{\partial u_t} = P_t u_t + B_t^*\sigma_t + p_t, \qquad\qquad t = 1, 2, \ldots, N. \end{aligned} \tag{3.3.31}$$

So the discrete Euler-Lagrange equations from the above are

$$\begin{aligned} x_t &= A_t x_{t-1} - B_t P_t^{-1} B_t^* \sigma_t + b_t - B_t P_t^{-1} p_t, \qquad x_0 = b_0, \\ \sigma_t &= R_t x_t + A_{t+1}^* \sigma_{t+1} + c_t, \qquad\qquad \sigma_N = R_N x_N + c_N. \end{aligned} \tag{3.3.32}$$

Using the sweep method solves (3.3.32), i.e., let

$$\sigma_t = S_t x_t + h_t, \tag{3.3.33}$$

then (3.3.32) gives from (3.3.33) that

$$\begin{aligned} S_t x_t + h_t =& R_t x_t + A_{t+1}^* S_{t+1}(I + B_{t+1}K_{t+1})^{-1}(A_t x_t + b_{t+1} \\ & - B_{t+1}P_{t+1}^{-1}p_{t+1} - B_{t+1}P_{t+1}^{-1}B_{t+1}^* h_{t+1}) + A_{t+1}^* h_{t+1} + c_t \\ =& R_t x_t + A_{t+1}^*(S_{t+1}^{-1} + B_{t+1}P_{t+1}^{-1}B_{t+1}^*)^{-1}(A_{t+1}x_t \\ & + b_{t+1} - B_{t+1}P_{t+1}^{-1}p_{t+1} + S_{t+1}^{-1}h_{t+1}) + c_t. \end{aligned} \tag{3.3.34}$$

Since $x_t$ is variable, the coefficients of $x_t$ on both sides of (3.3.34) are equal. Therefore, we obtain the equation in (3.3.30) and the equation in (3.3.29) from (3.3.34). The terminal conditions with these two equations are from (3.3.33) and from the condition in (3.3.32).

Hence, if (3.3.30) has a solution $\{S_1, S_2, \ldots, S_N\}$, then (3.3.33) and the second equality in (3.3.31) gives the optimal control law of $(\mathcal{R}_2^d)$ in (3.3.27) with $h_t$ from the ODE in (3.3.29). □

From the proof of Theorem 3.3.5, we can easily see that the closed loop dynamics matrices of $(\mathcal{RP}^d)$, $(\mathcal{R}_1^d)$ and $(\mathcal{R}_2^d)$ are the same. They are just the transpose of the dynamics matrices of the equations in (3.3.28) and (3.3.29). The closed loop dynamics matrix here is

$$M_{t-1} = (I + B_t K_t)^{-1} A_t = S_t^{-1}(S_t^{-1} + B_t P_t^{-1} B_t^*)^{-1} A_t. \tag{3.3.35}$$

We may easily deduce the following relationships.

$$(S_t^{-1} + B_t P_t^{-1} B_t^*)^{-1} = S_t - S_t B_t (P_t + B_t^* S_t B_t)^{-1} B_t^* S_t,$$
$$(I + S_t B_t P_t^{-1} B_t^*)^{-1} = I - S_t B_t (P_t + B_t^* S_t B_t)^{-1} B_t^*,$$
$$(I + B_t P_t^{-1} B_t^* S_t)^{-1} = I - B_t (P_t + B_t^* S_t B_t)^{-1} B_t^* S_t.$$

## 3.4. Four Splitting Choices for Multistage Models

We know from Section 3.1, that solving the saddle point problem associated with $L(u, v)$ on $U \times V$ of the multistage models is equivalent to solving the primal problem $(\mathcal{P}_d)$ and the dual problem $(\mathcal{Q}_d)$. Now, let us transfer the saddle point problem of the multistage models in Section 3.1 into the form of the fundamental problem $(\mathcal{P}_0)$ and consider the influences of using the different splitting methods to solve it relative to different choices.

Let

$$L(u, v) = L_1(u, v) + L_2(u, v) \tag{3.4.1}$$

and

$$T = \begin{pmatrix} \nabla_u L + \mathcal{N}_U \\ -\nabla_v L + \mathcal{N}_V \end{pmatrix}, \quad T_1 = \begin{pmatrix} \nabla_u L_1 \\ -\nabla_v L_1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} \nabla_u L_2 + \mathcal{N}_U \\ -\nabla_v L_2 + \mathcal{N}_V \end{pmatrix}. \tag{3.4.2}$$

where $L(u, v)$, $U$ and $V$ are given by (3.1.5) and $-l(u, v)$ is always put into $L_1(u, v)$ for simplicity. Example 2.1.3 implies that $T$ is maximal monotone.

From Chapter 2, it is clear that the key, for applying the known splitting methods to solve the fundamental problem relative to the above $T$, is to know how to compute $(I + \lambda T_1)^{-1}(u, v)$ and $(I + \lambda T_2)^{-1}(u, v)$, except the basic forward-backward splitting method for splitting $T = T_1 + T_2$. Therefore, it is desirable to compare, from computational aspects, the BFBS method with the others. Here we shall give four choices for splitting $L$ into $L_1$ and $L_2$, as examples, and transform the original problem into some simpler subproblems.

**Choice I. Basic Splitting:** Let $L = L_1 + L_2$ in (3.4.1) with

$$L_1(u, v) = -l(u, v); \qquad L_2(u, v) = L(u, v) - L_1(u, v). \tag{3.4.3}$$

Then $T_2 = \hat{T}_2 + \mathcal{N}_U \times \mathcal{N}_V$,

$$T_1(u,v) = \begin{pmatrix} -B_0^* y_1 \\ -B_1^* y_2 \\ \vdots \\ -B_N^* y_{N+1} \\ C_1 x_0 \\ \vdots \\ C_N x_{N-1} \\ C_{N+1} x_N \end{pmatrix}, \quad \hat{T}_2(u,v) = \begin{pmatrix} P_0 u_0 + p_0 \\ P_1 u_1 + p_1 - D_1^* v_1 \\ \vdots \\ P_N u_N + p_N - D_N^* v_N \\ Q_1 v_1 - q_1 + D_1 u_1 \\ \vdots \\ Q_N v_N - q_N + D_N u_N \\ Q_{N+1} v_{N+1} - q_{N+1} \end{pmatrix}. \tag{3.4.4}$$

Example 2.1.3 implies that $T_1$ and $T_2$ are maximal monotone.

For given $\bar{w} = (\bar{u}, \bar{v})$, we want to compute $w_1 = (I + \lambda T_1)^{-1}(\bar{w})$ and $w_2 = (I + \lambda T_2)^{-1}(\bar{w})$. Proposition 2.5.2 implies that $w_1 = (I + \lambda T_1)^{-1}(\bar{w})$ if and only if

$$w_1 = (u', v') = \operatorname{argminimax} \Lambda_1(w) \tag{3.4.5}$$

and $w_2 = (I + \lambda T_2)^{-1}(\bar{w})$ if and only if

$$w_2 = (u'', v'') = \operatorname*{argminimax}_{w \in U \times V} \Lambda_2(w), \tag{3.4.6}$$

where $w = (u, v)$ and

$$\begin{aligned} \Lambda_1(w) &= L_1(w) + \frac{1}{2\lambda}[\|u - \bar{u}\|^2 - \|v - \bar{v}\|^2] \\ \Lambda_2(w) &= L_2(w) + \frac{1}{2\lambda}[\|u - \bar{u}\|^2 - \|v - \bar{v}\|^2] \end{aligned} \tag{3.4.7}$$

Here $\|u\|^2 = u^* u = u \cdot u$ and so on.

(3.4.5) is a saddle point problem associated by two dynamical systems but without constraints. So we may consider its dual and primal problems.

**Theorem 3.4.1.** *The primal problem* $(\mathcal{P}_d^1)$ *and the dual problem* $(\mathcal{Q}_d^1)$ *of (3.4.5) are as follows:*

*minimize subject to (3.1.2):*

$(\mathcal{P}_d^1)$

$$f_1(u) = \frac{1}{2\lambda}\|u - \bar{u}\|^2 + \sum_{t=1}^{N+1} x_{t-1} \cdot [\frac{\lambda}{2} C_t^* C_t x_{t-1} - \bar{c}_t],$$

*and*

*maximize subject to (3.1.7):*

$$(\mathcal{Q}_d^1) \qquad g_1(v) = -\frac{1}{2\lambda}\|v-\bar{v}\|^2 - \sum_{t=0}^{N} y_{t+1}\cdot[\frac{\lambda}{2}B_tB_t^* y_{t+1} + \bar{b}_t],$$

*where* $\bar{c}_t = c_t + C_t^*\bar{v}_t$ *for* $t = 1,\ldots,N,N+1$ *and* $\bar{b}_t = b_t + B_t\bar{u}_t$ *for* $t = 0,1,\ldots,N$.

**Proof.** It is elementary from (3.4.3), (3.4.7) and (3.4.5). □

Now it may be seen that $(\mathcal{P}_d^1)$ and $(\mathcal{Q}_d^1)$ are two regulator problems and they are equivalent to the related regulator problem $(\mathcal{R}_2^d)$ in Section 3.3. In general, with time-dependent, they are not easy to solve.

(3.4.6) is merely a pure saddle point problem that may be decomposed into $N+2$ subproblems.

**Theorem 3.4.2.** *(3.4.6) is equivalent to the following subproblems:*

$$u_0'' = \operatorname*{argmin}_{u_0\in U_0}\left[p_0\cdot u_0 + \tfrac{1}{2}u_0\cdot P_0u_0 + \frac{1}{2\lambda}\|u_0-\bar{u}_0\|^2\right], \tag{3.4.8}$$

$$v_{N+1}'' = \operatorname*{argmin}_{s\in V_{N+1}}\left[\tfrac{1}{2}s\cdot Q_{N+1}s + \frac{1}{2\lambda}\|s-\bar{v}_{N+1}\|^2 - q_{N+1}\cdot s\right], \tag{3.4.9}$$

*and*

$$(u_t'', v_t'') = \operatorname*{argminimax}_{u\in U_t, v\in V_t} J_t(u,v), \qquad t = 1,\ldots,N. \tag{3.4.10}$$

*Where*

$$\begin{aligned} J_t(u,v) =& p_t\cdot u + \tfrac{1}{2}u\cdot P_tu + q_t\cdot v - \tfrac{1}{2}v\cdot Q_tv - v\cdot D_tu \\ &+ \frac{1}{2\lambda}[\|u-\bar{u}_t\|^2 - \|v-\bar{v}_t\|^2]. \end{aligned}$$

**Proof.** It is direct from (3.1.5), (3.4.1), (3.4.6) and (3.4.7). □

It is obvious from the above that solving $(I+\lambda T_2)^{-1}(w)$ is easier than computing $(I-\lambda T_2)(w)$ or $T_2(w)$, due to the normal cone operators, and obtaining $(I-\lambda T_1)(w)$ is much simpler than solving $(I+\lambda T_1)^{-1}(w)$.

**Choice II.** Let $L = L_1 + L_2$ in (3.4.1) with

$$L_1(u,v) = -\sum_{t=1}^{N} v_t\cdot D_tu_t - l(u,v); \qquad L_2(u,v) = L(u,v) - L_1(u,v). \tag{3.4.11}$$

Then $T_1$ and $T_2 = \hat{T}_2 + \mathcal{N}_U \times \mathcal{N}_V$ are given by

$$T_1(u,v) = \begin{pmatrix} -B_0^* y_1 \\ -D_1^* v_1 - B_1^* y_2 \\ \vdots \\ -D_N^* v_N - B_N^* y_{N+1} \\ D_1 u_1 + C_1 x_0 \\ \vdots \\ D_N u_N + C_N x_{N-1} \\ C_{N+1} x_N \end{pmatrix}, \hat{T}_2(u,v) = \begin{pmatrix} P_0 u_0 + p_0 \\ P_1 u_1 + p_1 \\ \vdots \\ P_N u_N + p_N \\ Q_1 v_1 - q_1 \\ \vdots \\ Q_N v_N - q_N \\ Q_{N+1} v_{N+1} - q_{N+1} \end{pmatrix} \tag{3.4.12}$$

and Example 2.1.3 implies that $T_1$ and $T_2$ are maximal monotone.

For given $\bar{w} = (\bar{u}, \bar{v})$, we also want to compute $w_1 = (I + \lambda T_1)^{-1}(\bar{w})$ and $w_2 = (I + \lambda T_2)^{-1}(\bar{w})$. Proposition 2.5.2 implies that $w_1 = (I + \lambda T_1)^{-1}(\bar{w})$ if and only if

$$w_1 = (u', v') = \operatorname{argminimax} \Lambda_1(w) \tag{3.4.13}$$

and $w_2 = (I + \lambda T_2)^{-1}(\bar{w})$ if and only if

$$w_2 = (u'', v'') = \operatorname*{argminimax}_{w \in U \times V} \Lambda_2(w), \tag{3.4.14}$$

where $w = (u, v)$ and

$$\begin{aligned} \Lambda_1(w) &= L_1(w) + \frac{1}{2\lambda}[\|u - \bar{u}\|^2 - \|v - \bar{v}\|^2] \\ \Lambda_2(w) &= L_2(w) + \frac{1}{2\lambda}[\|u - \bar{u}\|^2 - \|v - \bar{v}\|^2]. \end{aligned} \tag{3.4.15}$$

(3.4.13) is a saddle point problem, associated by two dynamical systems, but without constraints. So we may consider its dual and primal problems.

**Theorem 3.4.3.** *The primal problem $(\mathcal{P}_d^2)$ and the dual problem $(\mathcal{Q}_d^2)$ of (3.4.13) are as follows:*

*minimize subject to (3.1.2):*

$$(\mathcal{P}_d^2) \qquad \begin{aligned} f_2(u) = \sum_{t=1}^{N} [\frac{\lambda}{2} \|D_t u_t + C_t x_{t-1}\|^2 - \bar{v}_t \cdot D_t u_t] \\ + \frac{1}{2\lambda}\|u - \bar{u}\|^2 + \frac{\lambda}{2}\|C_{N+1} x_N\|^2 - \sum_{t=1}^{N+1} \bar{c}_t \cdot x_{t-1}, \end{aligned}$$

*and*

$$(\mathcal{Q}_d^2) \qquad \begin{aligned} &\textit{maximize subject to (3.1.7):} \\ g_2(v) = &- \sum_{t=1}^{N} [\frac{\lambda}{2} \|D_t^* v_t + B_t^* y_{t+1}\|^2 + v_t \cdot D_t \bar{u}_t] \\ &- \frac{1}{2\lambda}\|v - \bar{v}\|^2 - \sum_{t=0}^{N} \bar{b}_t \cdot y_{t+1} - \frac{\lambda}{2}\|B_0^* y_1\|^2, \end{aligned}$$

*where* $\bar{c}_t = c_t + C_t^* \bar{v}_t$ *for* $t = 1, \ldots, N, N+1$ *and* $\bar{b}_t = b_t + B_t \bar{u}_t$ *for* $t = 0, 1, \ldots, N$.

**Proof.** It is elementary from (3.4.11), (3.4.13), and (3.4.15). □

Now we may see that $(\mathcal{P}_d^2)$ and $(\mathcal{Q}_d^2)$ are two regulator problems and that they are equivalent to the related regulator problem $(\mathcal{R}_2^d)$ in Section 3.3. They are also hard to solve in general with time-dependent.

(3.4.14) is just a strongly convex quadratic problem and could be decomposed into $2N+2$ subproblems which are easily solved.

**Theorem 3.4.4.** *(3.4.14) is equivalent to the following subproblems:*

$$u_t'' = \underset{s \in U_t}{\operatorname{argmin}} \left[p_t \cdot s + \tfrac{1}{2} s \cdot P_t s + \frac{1}{2\lambda}\|s - \bar{u}_t\|^2\right], \quad t = 0, \ldots, N; \tag{3.4.16}$$

$$v_t'' = \underset{s \in V_t}{\operatorname{argmin}} \left[\tfrac{1}{2} s \cdot Q_t s + \frac{1}{2\lambda}\|s - \bar{v}_t\|^2 - q_t \cdot s\right], \quad t = 1, \ldots, N+1. \tag{3.4.17}$$

**Proof.** It is direct from (3.1.5), (3.4.12), (3.4.14), and (3.4.15). □

Similarly, it is obvious from the above that solving $(I + \lambda T_2)^{-1}(w)$ is easier than getting $(I - \lambda T_2)(w)$ or $T_2(w)$ due to the normal cone operators. Obtaining $(I - \lambda T_1)(w)$ is much simpler than solving $(I + \lambda T_1)^{-1}(w)$, on the other hand.

**Choice III. Projection splitting.** Let $L = L_1 + L_2$ in (3.4.1) with

$$L_2(u, v) = 0, L_1(u, v) = L(u, v). \tag{3.4.18}$$

Then $T_2 = \mathcal{N}_U \times \mathcal{N}_V$ and

$$T_1(u,v) = \begin{pmatrix} P_0u_0 + p_0 - B_0^*y_1 \\ P_1u_1 + p_1 - D_1^*v_1 - B_1^*y_2 \\ \vdots \\ P_Nu_N + p_N - D_N^*v_N - B_N^*y_{N+1} \\ Q_1v_1 - q_1 + D_1u_1 + C_1x_0 \\ \vdots \\ Q_Nv_N - q_N + D_Nu_N + C_Nx_{N-1} \\ Q_{N+1}v_{N+1} - q_{N+1} + C_{N+1}x_N \end{pmatrix}. \tag{3.4.19}$$

Example 2.1.3 implies that $T_1$ and $T_2$ are maximal monotone.

In this case, the basic FBS scheme is merely the following extended gradient projection:

$$(\bar{u},\bar{v}) = (I+\lambda T_2)^{-1}(I-\lambda T_1)(u,v) = \bar{P}_{U\times V}\big((u,v) - \lambda T_1(u,v)\big), \tag{3.4.20}$$

where $\bar{P}_{U\times V}$ represent the projection operator from $\mathbb{R}^k \times \mathbb{R}^l$ to $U \times V$.

For given $\bar{w} = (\bar{u},\bar{v})$, we still want to compute $w_1 = (I+\lambda T_1)^{-1}(\bar{w})$ and $w_2 = (I+\lambda T_2)^{-1}(\bar{w})$. Proposition 2.5.2 implies that $w_1 = (I+\lambda T_1)^{-1}(\bar{w})$ if and only if

$$w_1 = (u',v') = \operatorname{argminimax} \Lambda_1(w) \tag{3.4.21}$$

and $w_2 = (I+\lambda T_2)^{-1}(\bar{w})$ if and only if

$$w_2 = (u'',v'') = \operatorname*{argminimax}_{w\in U\times V} \Lambda_2(w), \tag{3.4.22}$$

where $w = (u,v)$ and

$$\begin{aligned} \Lambda_1(w) &= L(w) + \frac{1}{2\lambda}[\|u-\bar{u}\|^2 - \|v-\bar{v}\|^2] \\ \Lambda_2(w) &= \frac{1}{2\lambda}[\|u-\bar{u}\|^2 - \|v-\bar{v}\|^2]. \end{aligned} \tag{3.4.23}$$

(3.4.21) is a saddle point problem, associated by two dynamical systems, but without constraints. So we may consider its dual and primal problems as we did before.

**Theorem 3.4.5.** *The primal problem $(\mathcal{P}_d^3)$ and the dual problem $(\mathcal{Q}_d^3)$ of (3.4.21) are as follows:*

$$
(\mathcal{P}_d^3) \qquad
\begin{aligned}
&\textit{minimize subject to (3.1.2):}\\
f_3(u) =& \frac{1}{2\lambda}\sum_{t=1}^{N}[\bar v_t - \lambda(D_t u_t + C_t x_{t-1} - q_t)](I+\lambda Q_t)^{-1}\\
&[\bar v_t - \lambda(D_t u_t + C_t x_{t-1} - q_t)] + \frac{1}{2\lambda}\|u-\bar u\|^2 + \frac{1}{2\lambda}[\bar v_{N+1}\\
&+\lambda(q_{N+1} - C_{N+1}x_N)](I+\lambda Q_{N+1})^{-1}[\bar v_{N+1} + \lambda(q_{N+1} - C_{N+1}x_N)]\\
&+\sum_{t=0}^{N}(\tfrac{1}{2}u_t{\cdot}P_t u_t + p_t{\cdot}u_t - c_{t+1}{\cdot}x_t - \frac{1}{2\lambda}\|\bar v_{t+1}\|^2),
\end{aligned}
$$

*and*

$$
(\mathcal{Q}_d^3) \qquad
\begin{aligned}
&\textit{maximize subject to (3.1.7):}\\
g_3(v) =& -\frac{1}{2\lambda}\sum_{t=1}^{N}[\bar u_t + \lambda(D_t^* v_t + B_t^* y_{t+1} - p_t)](I+\lambda P_t)^{-1}\\
&[\bar u_t + \lambda(D_t^* v_t + B_t^* y_{t+1} - p_t)] - \frac{1}{2\lambda}\|v-\bar v\|^2\\
&-\frac{1}{2\lambda}[\bar u_0 + \lambda(B_0^* y_1 - p_0)](I+\lambda P_0)^{-1}[\bar u_0 + \lambda(B_0^* y_1 - p_0)]\\
&-\sum_{t=1}^{N+1}(\tfrac{1}{2}v_t{\cdot}Q_t v_t - q_t{\cdot}v_t + b_{t-1}{\cdot}y_t - \frac{1}{2\lambda}\|\bar u_{t-1}\|^2).
\end{aligned}
$$

**Proof.** It is elementary from (3.4.18), (3.4.21), and (3.4.23). ☐

Now it may still be seen that $(\mathcal{P}_d^3)$ and $(\mathcal{Q}_d^3)$ are two regulator problems and they could be transfered to the form of the problem $(\mathcal{R}_2^d)$ in Section 3.3.

(3.4.22) is also a strongly convex quadratic problem and may be decomposed into $2N+2$ subproblems which are just projections.

**Theorem 3.4.6.** *The solutions of (3.4.22) are given by the following projection operations:*

$$u_t'' = \bar P_{U_t}(\bar u_t), \qquad t = 0,\ldots,N; \tag{3.4.24}$$

$$v_t'' = \bar P_{V_t}(\bar v_t), \qquad t = 1,\ldots,N+1. \tag{3.4.25}$$

*Where $\bar{P}_{U_t}$ represents the projection operator from $\mathbb{R}^{k_t}$ to $U_t$ and similar to the others. That is,*

$$(I+\lambda T_2)^{-1} = \bar{P}_{U\times V} \quad \textit{for} \quad \lambda > 0. \tag{3.4.26}$$

**Proof.** It is direct from (3.1.5), (3.4.19), (3.4.22), (3.4.23) and from solving the strongly convex quadratic problems. □

Obviously, solving $(I+\lambda T_2)^{-1}(w)$ is much easier than getting $(I-\lambda T_2)(w)$ due to the normal cone operators and obtaining $(I-\lambda T_1)(w)$ is much simpler than solving $(I+\lambda T_1)^{-1}(w)$.

**Choice IV.** Let $L = L_1 + L_2$ in (3.4.1) with

$$\begin{aligned} L_2(u,v) &= \sum_{t=0}^{N} p_t \cdot u_t + \sum_{t=1}^{N+1} q_t \cdot v_t, \\ L_1(u,v) &= L(u,v) - L_2(u,v) \end{aligned} \tag{3.4.27}$$

Then $T_1$ and $T_2 = \hat{T}_2 + \mathcal{N}_U \times \mathcal{N}_V$ are given by

$$T_1(u,v) = \begin{pmatrix} P_0u_0 - B_0^* y_1 \\ P_1u_1 - D_1^*v_1 - B_1^*y_2 \\ \vdots \\ P_Nu_N - D_N^*v_N - B_N^*y_{N+1} \\ Q_1v_1 + D_1u_1 + C_1x_0 \\ \vdots \\ Q_Nv_N + D_Nu_N + C_Nx_{N-1} \\ Q_{N+1}v_{N+1} + C_{N+1}x_N \end{pmatrix}, \hat{T}_2(u,v) = \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_N \\ -q_1 \\ \vdots \\ -q_N \\ -q_{N+1} \end{pmatrix} \tag{3.4.28}$$

and Example 2.1.3 implies that $T_1$ and $T_2$ are maximal monotone.

For given $\bar{w} = (\bar{u}, \bar{v})$, we still want to compute $w_1 = (I+\lambda T_1)^{-1}(\bar{w})$ and $w_2 = (I+\lambda T_2)^{-1}(\bar{w})$. Proposition 2.5.2 implies that $w_1 = (I+\lambda T_1)^{-1}(\bar{w})$ if and only if

$$w_1 = (u', v') = \operatorname{argminimax} \Lambda_1(w) \tag{3.4.29}$$

and $w_2 = (I+\lambda T_2)^{-1}(\bar{w})$ if and only if

$$w_2 = (u'', v'') = \underset{w\in U\times V}{\operatorname{argminimax}} \Lambda_2(w), \tag{3.4.30}$$

where $w = (u, v)$ and

$$\begin{aligned}\Lambda_1(w) &= L_1(w) + \frac{1}{2\lambda}[\|u-\bar{u}\|^2 - \|v-\bar{v}\|^2]\\ \Lambda_2(w) &= L_2(w) + \frac{1}{2\lambda}[\|u-\bar{u}\|^2 - \|v-\bar{v}\|^2].\end{aligned} \tag{3.4.31}$$

(3.4.29) is a saddle point problem associated by two dynamical systems but without constraints. So we may consider its dual and primal problems as we did before.

**Theorem 3.4.7.** *The primal problem $(\mathcal{P}_d^4)$ and the dual problem $(\mathcal{Q}_d^4)$ of (3.4.29) are as follows:*

*minimize subject to (3.1.2):*

$$(\mathcal{P}_d^4)\quad \begin{aligned} f_4(u) =& \frac{1}{2\lambda}\sum_{t=1}^{N}[\bar{v}_t - \lambda(D_t u_t + C_t x_{t-1})](I+\lambda Q_t)^{-1}[\bar{v}_t\\ &- \lambda(D_t u_t + C_t x_{t-1})] + \frac{1}{2\lambda}\|u-\bar{u}\|^2\\ &+ \frac{1}{2\lambda}(\bar{v}_{N+1} - \lambda C_{N+1}x_N)(I+\lambda Q_{N+1})^{-1}(\bar{v}_{N+1} - \lambda C_{N+1}x_N)\\ &+ \sum_{t=0}^{N}(\tfrac{1}{2}u_t{\cdot}P_t u_t - c_{t+1}{\cdot}x_t - \frac{1}{2\lambda}\|\bar{v}_{t+1}\|^2),\end{aligned}$$

*and*

*maximize subject to (3.1.7):*

$$(\mathcal{Q}_d^4)\quad \begin{aligned} g_4(v) =& -\frac{1}{2\lambda}\sum_{t=1}^{N}[\bar{u}_t + \lambda(D_t^* v_t + B_t^* y_{t+1})](I+\lambda P_t)^{-1}[\bar{u}_t\\ &+ \lambda(D_t^* v_t + B_t^* y_{t+1})] - \frac{1}{2\lambda}\|v-\bar{v}\|^2\\ &- \frac{1}{2\lambda}(\bar{u}_0 + \lambda B_0^* y_1)(I+\lambda P_0)^{-1}(\bar{u}_0 + \lambda B_0^* y_1)\\ &- \sum_{t=1}^{N+1}(\tfrac{1}{2}v_t{\cdot}Q_t v_t + b_{t-1}{\cdot}y_t - \frac{1}{2\lambda}\|\bar{u}_{t-1}\|^2).\end{aligned}$$

**Proof.** It is elementary from (3.4.27), (3.4.29) and (3.4.31). □

Here we can still see that $(\mathcal{P}_d^4)$ and $(\mathcal{Q}_d^4)$ are two regulator problems and they could be transfered to the form of the problem $(\mathcal{R}_2^d)$ in Section 3.3.

(3.4.30) is also a strongly convex quadratic problem and could be decomposed into $2N+2$ subproblems which are readily solved.

**Theorem 3.4.8.** *The solutions of (3.4.30) are given by the following projection operations:*

$$u_t'' = \bar{P}_{U_t}(\bar{u}_t - \lambda p_t), \qquad t = 0, \ldots, N; \tag{3.4.32}$$

$$v_t'' = \bar{P}_{V_t}(\bar{v}_t + \lambda q_t), \qquad t = 1, \ldots, N+1. \tag{3.4.33}$$

*Where $\bar{P}_{U_t}$ and $\bar{P}_{V_t}$ represent the same projection operators as those in Choice III.*

**Proof.** It is direct from (3.1.5), (3.4.28), (3.4.30) and (3.4.31) and solving the strongly convex quadratic problems. □

Similarly from the above, solving $(I + \lambda T_2)^{-1}(w)$ is much easier than computing $(I - \lambda T_2)(w)$ or $T_2(w)$, due to the normal cone operators, and obtaining $(I - \lambda T_1)(w)$ is much simpler than solving $(I + \lambda T_1)^{-1}(w)$.

Therefore, from the perspective of computation, the suitable splitting methods for the multistage models in Section 3.1 seem those which only need to compute $(I + \lambda T_2)^{-1}(u, v)$ and $(I - \lambda T_1)(u, v)$ and/or $T_1(u, v)$ at each iteration. In summary, the basic FBS method seems best among the five kinds of splitting methods we mentioned in Section 2.6 from the view point of computation.

## 3.5. Four Splitting Choices for Continuous-Time Models

Similar to the last section, let us transfer the saddle point problem, associated with $\mathcal{J}(u, u_e; v, v_e)$ on $\mathcal{U} \times \mathcal{V}$ of the continuous-time models, into the form of Problem $(\mathcal{P}_0)$ now and consider the influence of applying the different splitting methods to solve it relative to different choices.

Let

$$T = \begin{pmatrix} \nabla_{(u,u_e)}\mathcal{J} + \mathcal{N}_{\mathcal{U}} \\ -\nabla_{(v,v_e)}\mathcal{J} + \mathcal{N}_{\mathcal{V}} \end{pmatrix}. \tag{3.5.1}$$

Example 2.1.3 implies that $T$ is a maximal monotone operator and Example 2.3.3 tells us that solving the saddle point problem associated with $\mathcal{J}(u, u_e; v, v_e)$ on $\mathcal{U} \times \mathcal{V}$ is equivalent to solving Problem $(\mathcal{P}_0)$ with $T$ given in (3.5.1).

For known splitting methods, it is easy to see that the key is to know how to compute $(I + \lambda T_2)^{-1}(u, u_e; v, v_e)$ and $(I + \lambda T_1)^{-1}(u, u_e; v, v_e)$ for given splitting

$T = T_1 + T_2$ except the BFBS method requiring $(I - \lambda T_1)(u, u_e; v, v_e)$ rather than $(I + \lambda T_1)^{-1}(u, u_e; v, v_e)$. Therefore, we also give four choices for $T$ splitting as examples and consider the computational comparison between the BFBS methods and the others.

**Choice I.** Let $T = T_1 + T_2$ from (3.5.1) with $T_2 = \hat{T}_2 + \mathcal{N}_{\mathcal{U}} \times \mathcal{N}_{\mathcal{V}}$ and

$$\hat{T}_2(u, u_e; v, v_e) = \begin{pmatrix} p(t) + P(t)u - D^*(t)v \\ p_e + P_e u_e - D_e^* v_e \\ Q(t)v + D(t)u - q(t) \\ Q_e v_e + D_e u_e - q_e \end{pmatrix}, \tag{3.5.2}$$

$$T_1(u, u_e; v, v_e) = \begin{pmatrix} -B^*(t)y(t) \\ -B_e^* y(t_0) \\ C(t)x(t) \\ C_e x(t_1) \end{pmatrix}. \tag{3.5.3}$$

In fact, if we let $z = (u, u_e; v, v_e)$ and $J'(z) = -[(u, u_e), (v, v_e)]$, $J''(z) = \mathcal{J}(z) - J'(z)$, then $T_1$ and $\hat{T}_2$ are just $T_{J'}$ and $T_{J''}$ respectively under the meaning of (2.1.10).

For given $\bar{z} = (\bar{u}, \bar{u}_e; \bar{v}, \bar{v}_e)$, we want to compute $z' = (I + \lambda T_1)^{-1}(\bar{z})$ and $z'' = (I + \lambda T_2)^{-1}(\bar{z})$. Proposition 2.5.2 implies that $z' = (I + \lambda T_1)^{-1}(\bar{z})$ if and only if

$$z' = (u', u'_e; v', v'_e) = \operatorname{argminimax} \Lambda_1(z) \tag{3.5.4}$$

and $z'' = (I + \lambda T_2)^{-1}(\bar{z})$ if and only if

$$z'' = (u'', u''_e; v'', v''_e) = \operatorname*{argminimax}_{z \in \mathcal{U} \times \mathcal{V}} \Lambda_2(z), \tag{3.5.5}$$

where

$$\begin{aligned} \Lambda_1(z) = J'(z) &+ \frac{1}{2\lambda} \int_{t_0}^{t_1} [\|u(t) - \bar{u}(t)\|^2 - \|v(t) - \bar{v}(t)\|^2] \, dt \\ &+ \frac{1}{2\lambda} [\|u_e - \bar{u}_e\|^2 - \|v_e - \bar{v}_e\|^2] \end{aligned} \tag{3.5.6}$$

$$\begin{aligned} \Lambda_2(z) = J''(z) &+ \frac{1}{2\lambda} \int_{t_0}^{t_1} [\|u(t) - \bar{u}(t)\|^2 - \|v(t) - \bar{v}(t)\|^2] \, dt \\ &+ \frac{1}{2\lambda} [\|u_e - \bar{u}_e\|^2 - \|v_e - \bar{v}_e\|^2]. \end{aligned} \tag{3.5.7}$$

Here $\|w\|^2$ stands for $w{\cdot}w$.

(3.5.5) is just a pure saddle point problem with positive definite matrices $P(t)+\frac{1}{2\lambda}I$, $P_e+\frac{1}{2\lambda}I$, $Q_e+\frac{1}{2\lambda}I$ and $Q(t)+\frac{1}{2\lambda}I$ and without state variables $x_t$ and $y_t$.

**Theorem 3.5.1.** *The optimality conditions of (3.5.5) for $t\in[t_0,t_1]$ a.e. are*

$$-\begin{pmatrix}\lambda^{-1}[u(t)-\bar{u}(t)]+P(t)u(t)+p(t)-D^*(t)v(t)\\ \lambda^{-1}[u_e-\bar{u}_e]+P_eu_e+p_e-D_e^*v_e\end{pmatrix}\in\mathcal{N}_{\mathcal{U}}(u(t),u_e), \tag{3.5.8}$$

$$-\begin{pmatrix}\lambda^{-1}[v(t)-\bar{v}(t)]+Q(t)v(t)-q(t)+D(t)u(t)\\ \lambda^{-1}[v_e-\bar{v}_e]+Q_ev_e-q_e+D_eu_e\end{pmatrix}\in\mathcal{N}_{\mathcal{V}}(v(t),v_e), \tag{3.5.9}$$

**Proof.** Example 2.1.3 and Example 2.3.3 imply that $z=(u,u_e;v,v_e)$ is a solution of (3.5.5) if and only if $0\in T_2(z)+T_0(z)$. Where

$$T_0(z)=\begin{pmatrix}u(t)-\bar{u}(t)\\ u_e-\bar{u}_e\\ v(t)-\bar{v}(t)\\ v_e-\bar{v}_e\end{pmatrix}. \tag{3.5.10}$$

Therefore, $z$ is the solution of (3.5.5) iff

$$-\hat{T}_2(z)-T_0(z)\in\mathcal{N}_{\mathcal{U}}(u(t),u_e)\times\mathcal{N}_{\mathcal{V}}(v(t),v_e)\qquad\text{for}\quad t\quad\text{a.e.}$$

The above is just (3.5.8) and (3.5.9). □

(3.5.4) is a saddle point problem associated by two dynamical systems but without constraints. So we may consider its dual and primal problems.

**Theorem 3.5.2.** *The primal problem $(\mathcal{P}_1^c)$ and the dual problem $(\mathcal{Q}_1^c)$ of (3.5.4) are as follows:*
*over the dynamical system (3.2.1),*

*minimize the functional*

$$(\mathcal{P}_1^c)\qquad \mathcal{F}_1(u,u_e)=\int_{t_0}^{t_1}\left[\frac{1}{2\lambda}\|u(t)-\bar{u}(t)\|^2+\frac{\lambda}{2}x(t){\cdot}C^*(t)C(t)x(t)\right.$$
$$\left.-\,\bar{c}(t){\cdot}x(t)\right]dt+\frac{1}{2\lambda}\|u_e-\bar{u}_e\|^2-\bar{c}_e{\cdot}x(t_1)+\frac{\lambda}{2}x(t_1){\cdot}C_e^*C_ex(t_1),$$

*and over the dynamical system (3.2.6),*

*maximize the functional*

$$(\mathcal{Q}_1^c) \qquad \mathcal{G}_1(v, v_e) = -\int_{t_0}^{t_1} \Big[\frac{1}{2\lambda}\|v(t) - \bar{v}(t)\|^2 + \frac{\lambda}{2} y(t) \cdot B(t) B^*(t) y(t) + \bar{b}(t) \cdot y(t)\Big]\, dt - [\frac{1}{2\lambda}\|v_e - \bar{v}_e\|^2 + \bar{b}_e \cdot y(t_0) + \frac{\lambda}{2} y(t_0) \cdot B_e B_e^* y(t_0)],$$

*where* $\bar{c}(t) = c(t) + C^*(t)\bar{v}(t)$, $\bar{c}_e = c_e + C_e^* \bar{v}_e$, $\bar{b}(t) = b(t) + B(t)\bar{u}(t)$ *and* $\bar{b}_e = b_e + B_e \bar{u}_e$.

**Proof.** Maximizing $\Lambda_1(u, u_e; v, v_e)$ for $(v, v_e)$ gives solution $(v, v_e)$ which depends on $(u, u_e)$. Then replacing the solution $(v, v_e)$ into $\Lambda_1(u, u_e; v, v_e)$ obtains $\mathcal{F}_1(u, u_e)$. The deducing process is elementary.

Similarly, minimizing $\Lambda_1(u, u_e; v, v_e)$ for $(u, u_e)$ will imply $\mathcal{G}_1(v, v_e)$. □

Now we may see that $(\mathcal{P}_1^c)$ and $(\mathcal{Q}_1^c)$ are two regulator problems and they are equivalent to the related regulator problem $(\mathcal{R}_4)$ in Section 3.3.

**Proposition 3.5.3.** *For given* $z = (u, u_e; v, v_e)$, *we have*

$$(I - \lambda T_1)(z) = \begin{pmatrix} u(t) + \lambda B^*(t) y(t) \\ u_e + \lambda B_e^* y(t_0) \\ v(t) - \lambda C(t) x(t) \\ v_e - \lambda C_e x(t_1) \end{pmatrix}, \tag{3.5.11}$$

$$(I - \lambda T_2)(z) = \begin{pmatrix} u - \lambda[p(t) + P(t)u - D^*(t)v] \\ u_e - \lambda[p_e + P_e u_e - D_e^* v_e] \\ v - \lambda[Q(t)v + D(t)u - q(t)] \\ v_e - \lambda[Q_e v_e + D_e u_e - q_e] \end{pmatrix} - \begin{pmatrix} \mathcal{N}_{\mathcal{U}}(u, u_e) \\ \mathcal{N}_{\mathcal{V}}(v, v_e) \end{pmatrix}, \tag{3.5.12}$$

*where* $x(t)$ *and* $y(t)$ *correspond to* $(u, u_e)$ *and* $(v, v_e)$ *respectively.*

**Proof.** They are direct from (3.5.2) and (3.5.3). □

It is elementary from the above that solving $(I + \lambda T_2)^{-1}(z)$ is easier than computing $(I - \lambda T_2)(z)$ or $T_2(z)$, due to the normal cone operators, and obtaining $(I - \lambda T_1)(z)$ is much simpler than solving $(I + \lambda T_1)^{-1}(z)$.

**Choice II.** Let $T = T_1 + T_2$ in (3.5.1) with

$$T_2(u, u_e; v, v_e) = \begin{pmatrix} p(t) + P(t)u \\ p_e + P_e u_e \\ Q(t)v - q(t) \\ Q_e v_e - q_e \end{pmatrix} + \begin{pmatrix} \mathcal{N}_{\mathcal{U}}(u, u_e) \\ \mathcal{N}_{\mathcal{V}}(v, v_e) \end{pmatrix},$$
$$T_1(u, u_e; v, v_e) = \begin{pmatrix} -D^*(t)v - B^*(t)y(t) \\ -D_e^* v_e - B_e^* y(t_0) \\ D(t)u + C(t)x(t) \\ D_e u_e + C_e x(t_1) \end{pmatrix}. \tag{3.5.13}$$

In fact, if we let $z = (u, u_e; v, v_e)$ and

$$J'(z) = -\int_{t_0}^{t_1} v \cdot D(t)u\, dt - v_e \cdot D_e u_e - [(u, u_e), (v, v_e)],$$
$$J''(z) = \mathcal{J}(z) - J'(z) + \delta_{\mathcal{U}}(u, u_e) - \delta_{\mathcal{V}}(v, v_e), \tag{3.5.14}$$

then $T_1$ and $T_2$ are just $T_{J'}$ and $T_{J''}$ respectively under the meaning of (2.1.10). Example 2.1.3 says that $T_1$ and $T_2$ are maximal monotone.

Similar to Choice I, for given $\bar{z} = (\bar{u}, \bar{u}_e; \bar{v}, \bar{v}_e)$, we want to compute $z' = (I + \lambda T_1)^{-1}(\bar{z})$ and $z'' = (I + \lambda T_2)^{-1}(\bar{z})$. Proposition 2.5.2 implies that $z' = (I + \lambda T_1)^{-1}(\bar{z})$ if and only if

$$z' = (u', u_e'; v', v_e') = \operatorname{argminimax} \Lambda_1(z) \tag{3.5.15}$$

and $z'' = (I + \lambda T_2)^{-1}(\bar{z})$ if and only if

$$z'' = (u'', u_e''; v'', v_e'') = \operatorname{argminimax} \Lambda_2(z), \tag{3.5.16}$$

where

$$\Lambda_1(z) = J'(z) + \frac{1}{2\lambda} \int_{t_0}^{t_1} [\|u(t) - \bar{u}(t)\|^2 - \|v(t) - \bar{v}(t)\|^2]\, dt$$
$$+ \frac{1}{2\lambda}[\|u_e - \bar{u}_e\|^2 - \|v_e - \bar{v}_e\|^2] \tag{3.5.17}$$

$$\Lambda_2(z) = J''(z) + \frac{1}{2\lambda} \int_{t_0}^{t_1} [\|u(t) - \bar{u}(t)\|^2 - \|v(t) - \bar{v}(t)\|^2]\, dt$$
$$+ \frac{1}{2\lambda}[\|u_e - \bar{u}_e\|^2 - \|v_e - \bar{v}_e\|^2]. \tag{3.5.18}$$

Here $J'$ and $J''$ are given in (3.5.14).

(3.5.16) is just four pure standard quadratic programming problems with positive definite matrices $P(t) + \frac{1}{2\lambda}I$, $P_e + \frac{1}{2\lambda}I$, $Q_e + \frac{1}{2\lambda}I$ and $Q(t) + \frac{1}{2\lambda}I$ and without state variables $x_t$ and $y_t$.

**Theorem 3.5.4.** *(3.5.16) is equivalent to the following subproblems:*

$$u'' = \operatorname*{argmin}_{u \in U^0} \int_{t_0}^{t_1} [p(t) \cdot u + \tfrac{1}{2} u \cdot P(t) u + \frac{1}{2\lambda} \|u - \bar{u}(t)\|^2] \, dt \tag{3.5.19}$$

$$u''_e = \operatorname*{argmin}_{u_e \in U_e} [p_e \cdot u_e + \tfrac{1}{2} u_e \cdot P_e u_e + \frac{1}{2\lambda} \|u_e - \bar{u}_e\|^2] \tag{3.5.20}$$

$$v'' = \operatorname*{argmin}_{v \in V^0} \int_{t_0}^{t_1} [\tfrac{1}{2} v \cdot Q(t) v - q(t) \cdot v + \frac{1}{2\lambda} \|v - \bar{v}(t)\|^2] \, dt \tag{3.5.21}$$

$$v''_e = \operatorname*{argmin}_{v_e \in V_e} [\tfrac{1}{2} v_e \cdot Q_e v_e - q_e \cdot v_e + \frac{1}{2\lambda} \|v_e - \bar{v}_e\|^2] \tag{3.5.22}$$

*where*

$$\begin{aligned} U^0 &= \{u \mid u \in \mathcal{L}^1, u(t) \in U(t) \quad \text{a.e. } \} \\ V^0 &= \{v \mid v \in \mathcal{L}^1, v(t) \in V(t) \quad \text{a.e. } \} \end{aligned} \tag{3.5.23}$$

**Proof.** They are elementary from (3.5.18), (3.5.16) and (3.2.5). □

**Corollary 3.5.5.** *$z''$ is the solution of (3.5.16) if and only if*

$$\begin{aligned} &- P(t)u'' - \lambda^{-1}[u'' - \bar{u}(t)] - p(t) \in \mathcal{N}_{U^0}(u''), \\ &- P_e u''_e - \lambda^{-1}[u''_e - \bar{u}_e] - p_e \in \mathcal{N}_{U_e}(u''_e), \\ &q(t) - Q(t)v'' - \lambda^{-1}[v'' - \bar{v}(t)] \in \mathcal{N}_{V^0}(v''), \\ &q_e - Q_e v''_e - \lambda^{-1}[v''_e - \bar{v}_e] \in \mathcal{N}_{V_e}(v''_e). \end{aligned} \tag{3.5.24}$$

**Proof.** These are the direct conclusion of Theorem 3.5.4 and Example 3.1.6.

□

(3.5.15) is a saddle point problem associated by two dynamical systems but without constraints. So we may consider its dual and primal problems.

**Theorem 3.5.6.** *The primal problem ($\mathcal{P}_2^c$) and the dual problem ($\mathcal{Q}_2^c$) of (3.5.15) are as follows:*

*over the dynamical system (3.2.1),*

*minimize the functional*

$$(\mathcal{P}_2^c) \qquad \begin{aligned} \mathcal{F}_2(u, u_e) = &\int_{t_0}^{t_1} \Big[\frac{1}{2\lambda} \|u(t) - \bar{u}(t)\|^2 - \bar{c}(t) \cdot x(t) + \frac{\lambda}{2} \|D(t)u(t) \\ &+ C(t)x(t)\|^2 - \bar{v}(t) \cdot D(t)u(t)\Big] \, dt + \frac{1}{2\lambda} \|u_e - \bar{u}_e\|^2 - \bar{c}_e \cdot x(t_1) \\ &+ \frac{\lambda}{2} \|D_e u_e + C_e x(t_1)\|^2 - \bar{v}_e \cdot D_e u_e, \end{aligned}$$

*and over the dynamical system (3.2.6),*

$$(\mathcal{Q}_2^c)\qquad \begin{aligned} &\text{maximize the functional}\\ &\mathcal{G}_2(v, v_e) = -\int_{t_0}^{t_1} \Big[\frac{1}{2\lambda}\|v(t) - \bar{v}(t)\|^2 + \bar{b}(t)\cdot y(t) + \frac{\lambda}{2}\|D^*(t)v(t)\\ &+ B^*(t)y(t)\|^2 + v(t)\cdot D(t)\bar{u}(t)\Big]\, dt - [\frac{1}{2\lambda}\|v_e - \bar{v}_e\|^2 + \bar{b}_e\cdot y(t_0)\\ &+ \frac{\lambda}{2}\|D_e^* v_e + B_e^* y(t_0)\|^2 + v_e\cdot D_e\bar{u}_e], \end{aligned}$$

*where* $\bar{c}(t) = c(t) + C^*(t)\bar{v}(t)$, $\bar{c}_e = c_e + C_e^*\bar{v}_e$, $\bar{b}(t) = b(t) + B(t)\bar{u}(t)$ *and* $\bar{b}_e = b_e + B_e\bar{u}_e$.

**Proof.** Similar to the proof of Theorem 3.5.2. □

Now we may see that $(\mathcal{P}_2^c)$ and $(\mathcal{Q}_2^c)$ are also two regulator problems and they can be transformed into the related standard regulator problem $(\mathcal{R}_4)$ in Section 3.3.

**Proposition 3.5.7.** *For given* $z = (u, u_e; v, v_e)$, *we have*

$$(I - \lambda T_1)(z) = \begin{pmatrix} u + \lambda[B^*(t)y + D^*(t)v] \\ u_e + \lambda[B_e^* y(t_0) + D_e^* v_e] \\ v - \lambda[C(t)x + D(t)u] \\ v_e - \lambda[C_e x(t_1) + D_e u_e] \end{pmatrix}, \tag{3.5.25}$$

$$(I - \lambda T_2)(z) = \begin{pmatrix} u - \lambda[p(t) + P(t)u + \mathcal{N}_{U^0}(u)] \\ u_e - \lambda[p_e + P_e u_e + \mathcal{N}_{U_e}(u_e)] \\ v - \lambda[Q(t)v + \mathcal{N}_{V^0}(v) - q(t)] \\ v_e - \lambda[Q_e v_e + \mathcal{N}_{V_e}(v_e) - q_e] \end{pmatrix}, \tag{3.5.26}$$

*where* $x$ *and* $y$ *correspond to* $(u, u_e)$ *and* $(v, v_e)$ *respectively.*

**Proof.** They are direct from (3.5.13). □

Similarly, it is obvious from the above that solving $(I + \lambda T_2)^{-1}(z)$ is easier than getting $(I - \lambda T_2)(z)$ or $T_2(z)$ due to the normal cone operators. Obtaining $(I - \lambda T_1)(z)$ is much simpler than solving $(I + \lambda T_1)^{-1}(z)$, on the other hand.

**Choice III.** Let $T = T_1 + T_2$ in (3.5.1) with

$$T_2(u, u_e; v, v_e) = \begin{pmatrix} p(t) + \mathcal{N}_{U^0}(u) \\ p_e + \mathcal{N}_{U_e}(u_e) \\ -q(t) + \mathcal{N}_{V^0}(v) \\ -q_e + \mathcal{N}_{V_e}(v_e) \end{pmatrix},$$
$$T_1(u, u_e; v, v_e) = \begin{pmatrix} P(t)u - D^*(t)v - B^*(t)y(t) \\ P_e u_e - D_e^* v_e - B_e^* y(t_0) \\ Q(t)v + D(t)u + C(t)x(t) \\ Q_e v_e + D_e u_e + C_e x(t_1) \end{pmatrix}. \tag{3.5.27}$$

In fact, if we let $z = (u, u_e; v, v_e)$ and

$$\begin{aligned} J'(z) = &\int_{t_0}^{t_1} [\tfrac{1}{2} u \cdot P(t) u - \tfrac{1}{2} v \cdot Q(t) v - v \cdot D(t) u]\, dt - v_e \cdot D_e u_e \\ &+ \tfrac{1}{2} u_e \cdot P_e u_e - \tfrac{1}{2} v_e \cdot Q_e v_e - [(u, u_e), (v, v_e)], \\ J''(z) = &\mathcal{J}(z) - J'(z) + \delta_{\mathcal{U}}(u, u_e) - \delta_{\mathcal{V}}(v, v_e), \end{aligned} \tag{3.5.28}$$

then $T_1$ and $T_2$ are just $T_{J'}$ and $T_{J''}$ respectively under the meaning of (2.1.10).

Similar to Choice I, for given $\bar{z} = (\bar{u}, \bar{u}_e; \bar{v}, \bar{v}_e)$, we want to compute $z' = (I + \lambda T_1)^{-1}(\bar{z})$ and $z'' = (I + \lambda T_2)^{-1}(\bar{z})$. Proposition 2.5.2 implies that $z' = (I + \lambda T_1)^{-1}(\bar{z})$ if and only if

$$z' = (u', u_e'; v', v_e') = \operatorname{argminimax} \Lambda_1(z) \tag{3.5.29}$$

and $z'' = (I + \lambda T_2)^{-1}(\bar{z})$ if and only if

$$z'' = (u'', u_e''; v'', v_e'') = \operatorname{argminimax} \Lambda_2(z), \tag{3.5.30}$$

where

$$\begin{aligned} \Lambda_1(z) = J'(z) &+ \frac{1}{2\lambda} \int_{t_0}^{t_1} [\|u(t) - \bar{u}(t)\|^2 - \|v(t) - \bar{v}(t)\|^2]\, dt \\ &+ \frac{1}{2\lambda} [\|u_e - \bar{u}_e\|^2 - \|v_e - \bar{v}_e\|^2] \\ \Lambda_2(z) = J''(z) &+ \frac{1}{2\lambda} \int_{t_0}^{t_1} [\|u(t) - \bar{u}(t)\|^2 - \|v(t) - \bar{v}(t)\|^2]\, dt \\ &+ \frac{1}{2\lambda} [\|u_e - \bar{u}_e\|^2 - \|v_e - \bar{v}_e\|^2]. \end{aligned}$$

Here $J'$ and $J''$ are given in (3.5.28).

(3.5.30) is just four pure standard quadratic programming problems with the same positive definite matrix $\frac{1}{2\lambda} I$ and without state variables $x_t$ and $y_t$.

**Theorem 3.5.8.** *(3.5.30) is equivalent to the following subproblems:*

$$u'' = \operatorname*{argmin}_{u \in U^0} \int_{t_0}^{t_1} [p(t) \cdot u + \frac{1}{2\lambda} \|u - \bar{u}(t)\|^2] \, dt \tag{3.5.31}$$

$$u_e'' = \operatorname*{argmin}_{u_e \in U_e} [p_e \cdot u_e + \frac{1}{2\lambda} \|u_e - \bar{u}_e\|^2] \tag{3.5.32}$$

$$v'' = \operatorname*{argmin}_{v \in V^0} \int_{t_0}^{t_1} [-q(t) \cdot v + \frac{1}{2\lambda} \|v - \bar{v}(t)\|^2] \, dt \tag{3.5.33}$$

$$v_e'' = \operatorname*{argmin}_{v_e \in V_e} [-q_e \cdot v_e + \frac{1}{2\lambda} \|v_e - \bar{v}_e\|^2] \tag{3.5.34}$$

*where $U^0$ and $V^0$ are defined by (3.5.23).*

**Proof.** They are elementary from (3.5.30) and (3.2.5). □

**Corollary 3.5.9.** *The solution $z''$ of (3.5.30) is given by*

$$\begin{aligned} u'' &= \bar{P}_{U^0}\big(\bar{u}(t) - \lambda p(t)\big), \\ u_e'' &= \bar{P}_{U_e}(\bar{u}_e - \lambda p_e), \\ v'' &= \bar{P}_{V^0}\big(\bar{v}(t) + \lambda q(t)\big), \\ v_e'' &= \bar{P}_{V_e}(\bar{v}_e + \lambda q_e). \end{aligned} \tag{3.5.35}$$

*Where $\bar{P}_{U^0}$ and $\bar{P}_{U_e}$ represent the projection operators from $\mathcal{L}^1\big([t_0, t_1], \mathbb{R}^k\big)$ to $U^0$ and from $\mathbb{R}^{k_e}$ to $U_e$ respectively. Similar definitions to $\bar{P}_{V^0}$ and $\bar{P}_{V_e}$.*

**Proof.** Solving the subproblems in Theorem 3.5.8 directly gives the results in (3.5.35). □

(3.5.29) is a saddle point problem associated by two dynamical systems but without constraints. So we may consider its dual and primal problems.

**Theorem 3.5.10.** *The primal problem $(\mathcal{P}_3^c)$ and the dual problem $(\mathcal{Q}_3^c)$ of (3.5.29)* *are as follows:*

*minimize the functional over the dynamical system (3.2.1)*

$$(\mathcal{P}_3^c)\qquad \begin{aligned} \mathcal{F}_3(u,u_e) = \int_{t_0}^{t_1} \Big\{ &\frac{1}{2\lambda}\big[\bar v(t) - \lambda D(t)u - \lambda C(t)x(t)\big]\cdot[I+\lambda Q(t)]^{-1}\big[\bar v(t) \\ &- \lambda C(t)x(t) - \lambda D(t)u\big] + \frac{1}{2\lambda}\|u-\bar u(t)\|^2 + \tfrac{1}{2}u\cdot P(t)u - \frac{1}{2\lambda}\|\bar v(t)\|^2 \\ &- c(t)\cdot x(t)\Big\}\, dt + \frac{1}{2\lambda}\|u_e - \bar u_e\|^2 + \tfrac12 u_e\cdot P_e u_e - \frac{1}{2\lambda}\|\bar v_e\|^2 - c_e\cdot x(t_1) \\ &+ \frac{1}{2\lambda}[\bar v_e - \lambda D_e u_e - \lambda C_e x(t_1)]\cdot(I+\lambda Q_e)^{-1}[\bar v_e - \lambda D_e u_e - \lambda C_e x(t_1)] \end{aligned}$$

*and*

*maximize the functional over the dynamical system (3.2.6)*

$$(\mathcal{Q}_3^c)\qquad \begin{aligned} \mathcal{G}_3(v,v_e) = -\int_{t_0}^{t_1} \{ &\frac{1}{2\lambda}\|v-\bar v(t)\|^2 + \tfrac12 v\cdot Q(t)v + b(t)\cdot y(t) - \frac{1}{2\lambda}\|\bar u(t)\|^2 \\ &+ \frac{1}{2\lambda}[\bar u(t) + \lambda D^*(t)v + \lambda B^*(t)y(t)]\cdot[I+\lambda P(t)]^{-1}[\bar u(t) + \lambda D^*(t)v \\ &+ \lambda B^*(t)y(t)]\}\, dt - [\frac{1}{2\lambda}\|v_e - \bar v_e\|^2 + b_e\cdot y(t_0) + \tfrac12 v_e\cdot Q_e v_e - \frac{1}{2\lambda}\|\bar u_e\|^2] \\ &- \frac{1}{2\lambda}[\bar u_e + \lambda D_e^* v_e + \lambda B_e^* y(t_0)]\cdot(I+\lambda P_e)^{-1}[\bar u_e + \lambda D_e^* v_e + \lambda B_e^* y(t_0)] \end{aligned}$$

**Proof.** Similar to the proof of Theorem 3.5.2. □

Now we may see that $(\mathcal{P}_3^c)$ and $(\mathcal{Q}_3^c)$ are also two regulator problems and they can be transformed into the related standard regulator problem $(\mathcal{R}_4)$ in Section 3.3.

**Proposition 3.5.11.** *For given* $z = (u, u_e; v, v_e)$, *we have*

$$(I-\lambda T_1)(z) = \begin{pmatrix} u + \lambda[B^*(t)y + D^*(t)v - P(t)u] \\ u_e + \lambda[B_e^* y(t_0) + D_e^* v_e - P_e u_e] \\ v - \lambda[C(t)x + D(t)u + Q(t)v] \\ v_e - \lambda[C_e x(t_1) + D_e u_e + Q_e v_e] \end{pmatrix}, \tag{3.5.36}$$

$$(I-\lambda T_2)(z) = \begin{pmatrix} u - \lambda[p(t) + \mathcal{N}_{U^0}(u)] \\ u_e - \lambda[p_e + \mathcal{N}_{U_e}(u_e)] \\ v - \lambda[\mathcal{N}_{V^0}(v) - q(t)] \\ v_e - \lambda[\mathcal{N}_{V_e}(v_e) - q_e] \end{pmatrix}, \tag{3.5.37}$$

*where $x$ and $y$ correspond to $(u, u_e)$ and $(v, v_e)$ respectively.*

**Proof.** They are direct from (3.5.27). □

Obviously, solving $(I+\lambda T_2)^{-1}(z)$ is much easier than getting $(I-\lambda T_2)(z)$ due to the normal cone operators and obtaining $(I-\lambda T_1)(z)$ is much simpler than solving $(I+\lambda T_1)^{-1}(z)$.

**Choice IV. Projection splitting.** Let $T = T_1 + T_2$ in (3.5.1) with

$$T_1 = \begin{pmatrix} \nabla_{(u,u_e)}\mathcal{J} \\ -\nabla_{(v,v_e)}\mathcal{J} \end{pmatrix}, T_2 = \begin{pmatrix} \mathcal{N}_{\mathcal{U}} \\ \mathcal{N}_{\mathcal{V}} \end{pmatrix}. \tag{3.5.38}$$

That is,

$$T_1(u, u_e; v, v_e) = \begin{pmatrix} P(t)u - D^*(t)v - B^*(t)y(t) + p(t) \\ P_e u_e - D_e^* v_e - B_e^* y(t_0) + p_e \\ Q(t)v + D(t)u + C(t)x(t) - q(t) \\ Q_e v_e + D_e u_e + C_e x(t_1) - q_e \end{pmatrix}. \tag{3.5.39}$$

$T_1$ and $T_2$ are maximal monotone from Example 2.1.3.

Similar to Choice III, for given $\bar{z} = (\bar{u}, \bar{u}_e; \bar{v}, \bar{v}_e)$, we want to compute $z' = (I+\lambda T_1)^{-1}(\bar{z})$ and $z'' = (I+\lambda T_2)^{-1}(\bar{z})$. Proposition 2.5.2 implies that $z' = (I+\lambda T_1)^{-1}(\bar{z})$ if and only if

$$z' = (u', u_e'; v', v_e') = \operatorname{argminimax} \Lambda_1(z) \tag{3.5.40}$$

and $z'' = (I+\lambda T_2)^{-1}(\bar{z})$ if and only if

$$z'' = (u'', u_e''; v'', v_e'') = \operatorname*{argminimax}_{z \in \mathcal{U}\times\mathcal{V}} \Lambda_2(z), \tag{3.5.41}$$

where

$$\begin{aligned} \Lambda_1(z) &= \mathcal{J}(z) + \frac{1}{2\lambda}\int_{t_0}^{t_1} [\|u(t)-\bar{u}(t)\|^2 - \|v(t)-\bar{v}(t)\|^2]\,dt \\ &\quad + \frac{1}{2\lambda}[\|u_e-\bar{u}_e\|^2 - \|v_e-\bar{v}_e\|^2] \\ \Lambda_2(z) &= +\frac{1}{2\lambda}\int_{t_0}^{t_1} [\|u(t)-\bar{u}(t)\|^2 - \|v(t)-\bar{v}(t)\|^2]\,dt \\ &\quad + \frac{1}{2\lambda}[\|u_e-\bar{u}_e\|^2 - \|v_e-\bar{v}_e\|^2]. \end{aligned}$$

(3.5.41) is just four pure standard quadratic programming problems with the same positive definite matrix $\frac{1}{2\lambda}I$ and without state variables $x_t$ and $y_t$.

**Theorem 3.5.12.** *(3.5.41) is equivalent to the following subproblems:*

$$u'' = \operatorname*{argmin}_{u \in U^0} \int_{t_0}^{t_1} \frac{1}{2\lambda} \|u - \bar{u}(t)\|^2 \, dt \tag{3.5.42}$$

$$u_e'' = \operatorname*{argmin}_{u_e \in U_e} \frac{1}{2\lambda} \|u_e - \bar{u}_e\|^2 \tag{3.5.43}$$

$$v'' = \operatorname*{argmin}_{v \in V^0} \int_{t_0}^{t_1} \frac{1}{2\lambda} \|v - \bar{v}(t)\|^2 \, dt \tag{3.5.44}$$

$$v_e'' = \operatorname*{argmin}_{v_e \in V_e} \frac{1}{2\lambda} \|v_e - \bar{v}_e\|^2] \tag{3.5.45}$$

*where $U^0$ and $V^0$ are defined by (3.5.23).*

**Proof.** They are elementary from (3.5.41) and (3.2.5). □

**Corollary 3.5.13.** *The solution $z''$ of (3.5.41) is given by*

$$\begin{aligned} u'' &= \bar{P}_{U^0}(\bar{u}), & u_e'' &= \bar{P}_{U_e}(\bar{u}_e), \\ v'' &= \bar{P}_{V^0}(\bar{v}), & v_e'' &= \bar{P}_{V_e}(\bar{v}_e). \end{aligned} \tag{3.5.46}$$

*Therefore,*

$$z'' = \bar{P}_{\mathcal{U} \times \mathcal{V}}(\bar{z}),$$

$$(I + \lambda T_2)^{-1} = \bar{P}_{\mathcal{U} \times \mathcal{V}} \quad \text{for} \quad \lambda > 0. \tag{3.5.47}$$

*Where $\bar{P}$ always represents a suitable projection operator similar to those in Corollary 3.5.9.*

**Proof.** Solving the subproblems in Theorem 3.5.12 directly gives the results in (3.5.46). □

(3.5.40) is a saddle point problem associated by two dynamical systems but without constraints. So we may consider its dual and primal problems.

**Theorem 3.5.14.** *The primal problem* $(\mathcal{P}_4^c)$ *and the dual problem* $(\mathcal{Q}_4^c)$ *of (3.5.40) are as follows:*

*minimize the functional over the dynamical system (3.2.1)*

$$(\mathcal{P}_4^c)\qquad \begin{aligned} \mathcal{F}_4(u,u_e) = \int_{t_0}^{t_1} \Big\{ & \frac{1}{2\lambda}\Big[\bar{v}(t) - \lambda\big(D(t)u + C(t)x(t) - q(t)\big)\Big]\cdot[I+\lambda Q(t)]^{-1} \\ & \Big[\bar{v}(t) - \lambda\big(D(t)u + C(t)x(t) - q(t)\big)\Big] + \frac{1}{2\lambda}\|u-\bar{u}(t)\|^2 + \tfrac{1}{2}u\cdot P(t)u \\ & + p(t)\cdot u - c(t)\cdot x(t) - \frac{1}{2\lambda}\|\bar{v}(t)\|^2\Big\}\, dt + p_e\cdot u_e \\ & + \frac{1}{2\lambda}\|u_e - \bar{u}_e\|^2 + \tfrac{1}{2}u_e\cdot P_e u_e - \frac{1}{2\lambda}\|\bar{v}_e\|^2 - c_e\cdot x(t_1) \\ & + \frac{1}{2\lambda}\big[\bar{v}_e - \lambda\big(D_e u_e + C_e x(t_1) - q_e\big)\big]\cdot(I+\lambda Q_e)^{-1}\big[\bar{v}_e \\ & - \lambda\big(D_e u_e + C_e x(t_1) - q_e\big)\big] \end{aligned}$$

*and*

*maximize the functional over the dynamical system (3.2.6)*

$$(\mathcal{Q}_4^c)\qquad \begin{aligned} \mathcal{G}_4(v,v_e) = -\int_{t_0}^{t_1} \{ & \frac{1}{2\lambda}\|v-\bar{v}(t)\|^2 + \tfrac{1}{2}v\cdot Q(t)v + b(t)\cdot y(t) - \frac{1}{2\lambda}\|\bar{u}(t)\|^2 \\ & + \frac{1}{2\lambda}\big[\bar{u}(t) + \lambda\big(D^*(t)v + B^*(t)y(t) - p(t)\big)\big]\cdot[I+\lambda P(t)]^{-1}\big[\bar{u}(t) \\ & + \lambda\big(D^*(t)v + B^*(t)y(t) - p(t)\big)\big] - q(t)\cdot v\}\, dt + q_e\cdot v_e - b_e\cdot y(t_0) \\ & - [\frac{1}{2\lambda}\|v_e - \bar{v}_e\|^2 + \tfrac{1}{2}v_e\cdot Q_e v_e - \frac{1}{2\lambda}\|\bar{u}_e\|^2] - \frac{1}{2\lambda}\big[\bar{u}_e + \lambda\big(D_e^* v_e \\ & + B_e^* y(t_0) - p_e\big)\big]\cdot(I+\lambda P_e)^{-1}\big[\bar{u}_e + \lambda\big(D_e^* v_e + B_e^* y(t_0) - p_e\big)\big] \end{aligned}$$

**Proof.** Similar to the proof of Theorem 3.5.2. ☐

Now we may see that $(\mathcal{P}_4^c)$ and $(\mathcal{Q}_4^c)$ are also two regulator problems and they can be transformed into the related standard regulator problem $(\mathcal{R}_4)$ in Section 3.3.

**Proposition 3.5.15.** *For given* $z = (u, u_e; v, v_e)$, *we have*

$$(I - \lambda T_1)(z) = \begin{pmatrix} u + \lambda[B^*(t)y + D^*(t)v - P(t)u - p(t)] \\ u_e + \lambda[B_e^* y(t_0) + D_e^* v_e - P_e u_e - p_e] \\ v - \lambda[C(t)x + D(t)u + Q(t)v - q(t)] \\ v_e - \lambda[C_e x(t_1) + D_e u_e + Q_e v_e - q_e] \end{pmatrix}, \qquad (3.5.48)$$

$$(I - \lambda T_2)(z) = \begin{pmatrix} u - \mathcal{N}_{U^0}(u) \\ u_e - \mathcal{N}_{U_e}(u_e) \\ v - \mathcal{N}_{V^0}(v) \\ v_e - \mathcal{N}_{V_e}(v_e) \end{pmatrix}, \tag{3.5.49}$$

*where $x$ and $y$ correspond to $(u, u_e)$ and $(v, v_e)$ respectively.*

**Proof.** They are direct from (3.5.38) and (3.5.39). □

In this case, the basic FBS scheme is just the following extended gradient projection:

$$\bar{z} = (I + \lambda T_2)^{-1}(I - \lambda T_1)(z) = \bar{P}_{\mathcal{U}\times\mathcal{V}}\big(z - \lambda T_1(z)\big). \tag{3.5.50}$$

Similarly from the above, solving $(I + \lambda T_2)^{-1}(w)$ is much easier than computing $(I - \lambda T_2)(w)$ or $T_2(w)$, due to the normal cone operators, and obtaining $(I - \lambda T_1)(w)$ is much simpler than solving $(I + \lambda T_1)^{-1}(w)$.

Therefore, from the perspective of computation, the suitable splitting methods for the multistage models in Section 3.1 seem those which only need to compute $(I + \lambda T_2)^{-1}(z)$ and $(I - \lambda T_1)(z)$ and/or $T_1(z)$ at each iteration. In summary, the basic FBS method seems best among the five kinds of splitting methods we mentioned in Section 2.6 from the view point of computation.