Chapter 5

EXTENDED FORWARD-BACKWARD SPLITTING METHODS AND CONVERGENCE

This chapter extends the forward-backward splitting method for the problem that seeks to find a zero point of a maximal monotone operator and discusses its convergence properties. Under the extension, the new method possesses more flexibility and may obtain better convergent results than the original one. A local quadratic convergence theorem and a global superlinear convergence theorem are proved in Section 5.6. Since we shall deal with set and graphical convergence in the discussion of this chapter, we shall introduce some basic concepts of those convergence in the first section.

5.1. Set and Graphical Limits and Convergence

Before we go to extended FBS methods, we shall give concepts of set limits and set convergence here. For more details about these concepts and properties, one may refer to the related lecture notes of Rockafellar.

Definition 5.1.1 (inner and outer limits). *For a sequence $\{C^k\}_{k=0}^{\infty}$ of subsets of $\mathcal{H}$, the inner limit is the set*

$$\liminf_{k\to\infty} C^k := \{x \mid \exists x^k \in C^k \quad \text{for all} \quad k \geq 0, \quad \text{with} \quad \lim_{k\to\infty} x^k = x\}, \tag{5.1.1}$$

while the outer limit is the set

$$\limsup_{k\to\infty} C^k := \{x \mid \exists x^{k_i} \in C^{k_i} \quad \text{for} \quad i \geq 0, \quad \text{with} \quad \lim_{i\to\infty} x^{k_i} = x\}, \tag{5.1.2}$$

where $\{C^{k_i}\}$ is an any subsequence of $\{C^k\}$. The limit of the sequence exists if the inner and outer limit sets are equal:

$$\lim_{k\to\infty} C^k := \liminf_{k\to\infty} C^k = \limsup_{k\to\infty} C^k. \tag{5.1.3}$$

Although the inner and outer limits of a sequence $\{C^k\}$ always exist, in contrast to the limit itself, they may sometimes be $\emptyset$. When $C^k \neq \emptyset$ for all k, the set $\liminf_k C^k$ consists of all possible limit points of sequences $\{x^k\}$, with $x^k \in C^k$ for all $k \geq 0$, whereas, $\limsup_k C^k$ consists of all possible cluster points of such sequences. In any case, $\liminf_k C^k \subset \limsup_k C^k$.

Definition 5.1.2. *The set $\{C^k\}$ is said to converge to a set C when $\lim_k C^k$ exists in the sense of Definition 5.1.1 and equals C. We denote it as $C^k \to C$. This kind of set convergence is also known as ordinary, or Painlevé-Kuratowski, convergence.*

In the following, we give some properties about set limits and convergence, without proofs, since one can easily find them in Rockafellar's lecture notes for $\mathcal{H} = \mathbb{R}^n$ and extend them into a general Hilbert space.

Proposition 5.1.1 (closedness of limit sets). *For any sequence of sets C^k in $\mathcal{H}$, both the inner limit set $\liminf_k C^k$ and the outer limit set $\limsup_k C^k$ are closed. Furthermore, they depend only on the closures $\operatorname{cl} C^k$, in the sense that*

$$\operatorname{cl} C^k = \operatorname{cl} D^k \quad \Longrightarrow \quad \begin{cases} \liminf_k C^k = \liminf_k D^k \\ \limsup_k C^k = \limsup_k D^k. \end{cases}$$

Thus, whenever $\lim_k C^k$ exists, it is closed. (If $C^k \equiv C$, then $\lim_k C^k = \operatorname{cl} C$.)

Proposition 5.1.2 (limits of monotone sequences). *Monotone sequences $\{C^k\}$ always converge:*

(a) if C^k is nondecreasing (i.e., $C^{k+1} \supset C^k$), then $\lim_k C^k = \operatorname{cl} \bigcup_{k=0}^{\infty} C^k$;

(b) if C^k is nonincreasing (i.e., $C^{k+1} \subset C^k$), then $\lim_k C^k = \bigcap_{k=0}^{\infty} \operatorname{cl} C^k$.

Theorem 5.1.3 (uniformity in set convergence). *For subsets C^k and C of $\mathcal{H}$ with C closed, if $\mathbb{B}$ is that unit ball in (5.1.4), one has*

(a) $C \subset \liminf_k C^k$, if and only if, for every $\mu > 0$ and $\varepsilon > 0$ there is an index k_0 such that

$$C \cap \mu \mathbb{B} \subset C^k + \varepsilon \mathbb{B} \quad \text{for all} \quad k \geq k_0;$$

(b) $C \supset \limsup_k C^k$, if and only if, for every $\mu > 0$ and $\varepsilon > 0$ there is an index k_0 such that

$$C^k \cap \mu \mathbb{B} \subset C + \varepsilon \mathbb{B} \quad \text{for all} \quad k \geq k_0.$$

Thus, $C = \lim_k C^k$, if and only if, for every $\mu > 0$ and $\varepsilon > 0$ there is an index k_0 such that both of these inclusions hold.

Proposition 5.1.4 (limits of convex sets)**.** *Suppose the sequence $\{C^k\}$ consists of convex subsets of $\mathcal{H}$. Then, $\liminf_k C^k$ is convex, and so too is $\lim_k C^k$ when it exists. (But $\limsup_k C^k$ need not be convex in general.)*

Proposition 5.1.5 (limits of cones)**.** *For a sequence of cones C^k in $\mathcal{H}$, the inner and outer limits, as well as the limit if it exists, are cones. Moreover, $\limsup_k C^k \neq \{0\}$ when every $C^k \neq \{0\}$.*

Proposition 5.1.6 (preservation of set limits)**.** *Let F be a one-to-one mapping from $\mathcal{H}$ onto $\mathcal{H}$, such that F and F^{-1} are both continuous. Then*

$$F(\liminf_k C^k) = \liminf_k F(C^k), \quad F(\limsup_k C^k) = \limsup_k F(C^k),$$

hence, $C^k \to C$, if and only if, $F(C^k) \to F(C)$.

Now let us consider convergence of operators or multivalued mappings. For a sequence of operators $T^k : \mathcal{H} \rightrightarrows \mathcal{H}$, various forms of convergence may be developed. Here we only give pointwise convergence and graphical convergence.

Definition 5.1.3. *Let T and $\{T^k\}_{k=0}^{\infty}$ be operators over $\mathcal{H}$.*

(a) The sequence $\{T^k\}$ is said to converge pointwisely to T, if $T^k(z) \to T(z)$ for all $z \in \mathcal{H}$, in the sense of the set convergence in Definition 5.1.2. We take the notation of pointwise convergence as $T^k \to T$.

(b) Similarly, the (pointwise) inner limit $\liminf_k T^k$ and the (pointwise) outer limit $\limsup_k T^k$ of the sequence $\{T^k\}$ are given by

$$(\liminf_k T^k)(z) := \liminf_k T^k(z), \quad (\limsup_k T^k)(z) := \limsup_k T^k(z)$$

for all $z \in \mathcal{H}$. When these agree, the common operator is called the limit of the sequence and is denoted by $\lim_k T^k$.

Thus, $T^k \to T$, if and only if, $T = \lim_k T^k$.

Definition 5.1.4 (inner and outer graphical limits). *For any operator sequence* $\{T^k\}_{k=0}^{\infty}$ *over* $\mathcal{H}$,

(a) *the inner graphical limit, denoted by* $\text{g-}\liminf_k T^k$, *is the operator having its graph as the set* $\liminf_k(\operatorname{gph} T^k)$*:*

$$\operatorname{gph}(\text{g-}\liminf_k T^k) := \liminf_k(\operatorname{gph} T^k).$$

(b) *The outer graphical limit, denoted by* $\text{g-}\limsup_k T^k$, *is the operator having its graph as the set* $\limsup_k(\operatorname{gph} T^k)$*:*

$$\operatorname{gph}(\text{g-}\limsup_k T^k) := \limsup_k(\operatorname{gph} T^k).$$

(c) *When the inner and outer graphical limits are the same operator* T, *the graphical limit* $\text{g-}\lim_k T^k$ *is said to exist:* $\text{g-}\lim_k T^k := T$.

(d) *In the event of (c), the operator* T^k *is said to graph converge to* T, *a condition symbolized by* $T^k \xrightarrow{g} T$. *Thus,*

$$T^k \xrightarrow{g} T \iff \operatorname{gph} T^k \to \operatorname{gph} T.$$

Proposition 5.1.7 (formulas for graphical limits). *For an operator sequence* $\{T^k\}_{k=0}^{\infty}$ *over* $\mathcal{H}$, *one has*

$$(\text{g-}\liminf_k T^k)(z) = \{w \mid \exists z^k \to z \quad \textit{with} \quad w \in \liminf_{k\to\infty} T^k(z^k)\},$$
$$(\text{g-}\limsup_k T^k)(z) = \{w \mid \exists z^k \to z \quad \textit{with} \quad w \in \limsup_{k\to\infty} T^k(z^k)\}.$$

5.2. Extended Forward-Backward Splitting (EFBS) Methods

Let $\mathcal{H}$ be a real Hilbert space and T_1, T_2 be maximal monotone operators on $\mathcal{H}$, such that $T = T_1 + T_2$ is maximal. For a sequence $\{\lambda_k\}_{k=0}^{\infty}$ of positive scalars, a sequence $\{z^k\}_{k=0}^{\infty} \subset \mathcal{H}$ is said to be generated by the basic forward-backward scheme, in ordinary sense, if

$$z^{k+1} \in (I + \lambda_k T_2)^{-1}(I - \lambda_k T_1)(z^k), \forall k \geq 0. \tag{5.2.1}$$

The convergence of $\{z^k\}$ from (5.2.1) is given in the last chapter as follows.

BFBS Theorem. *Let $T = T_1 + T_2$, and $\operatorname{zer}(T) = T^{-1}(0) \neq \emptyset$. Suppose $\{z^k\}$ is generated through (5.2.1) for any starting point $z^0 \in \operatorname{dom}(T_1)$. Then*

(a) $\{z^k\}$ will weakly converge to a zero point of T under Condition D of Section 4.3.

(b) $\{z^k\}$ will strongly converge to the unique zero point of T with Q-linear rate if Condition A or Condition B or Condition C in Section 4.3 is satisfied.

Now let us generalize the above forward-backward scheme.

First, let us be reminded that an operator R on $\mathcal{H}$ is called a *bounded linear operator*, if

(a) R is single-valued and $\operatorname{dom}(R) = \mathcal{H}$.

(b) For any $x, y \in \mathcal{H}$ and $\lambda, \eta \in \mathbb{R}$, R satisfies the following

$$R(\lambda x + \eta y) = \lambda R(x) + \eta R(y).$$

(c) There is a constant $M \geq 0$, such that for any $x \in \mathcal{H}$

$$\|R(x)\| \leq M\|x\|.$$

We declare that P is the *adjoint* of R, if P is a bounded linear operator on $\mathcal{H}$, and for any $x, y \in \mathcal{H}$

$$\langle R(x), y \rangle = \langle x, P(y) \rangle.$$

We denote $P = R^*$. When $R = R^*$, R is said to be *symmetric*.

Recall that an operator T is *strictly monotone* on a set $Z \subset \mathcal{H}$ if

$$\langle T(z) - T(z'), z - z' \rangle > 0 \qquad \forall z, z' \in Z \quad \text{and} \quad z \neq z'.$$

In the following, we always assume that T_1, T_2, T_1^k, T_2^k are maximal monotone operators on $\mathcal{H}$, such that $T^k = T_1^k + T_2^k$, $T = T_1 + T_2$ are maximal for $k \geq 0$, and $\limsup_k \operatorname{zer}(T^k) \subset \operatorname{zer}(T)$.

Definition 5.2.1. *Let H_k be a continuous and strictly monotone operator for all $k \geq 0$. For a sequence $\{\lambda_k\}_{k=0}^{\infty}$ of positive scalars, a sequence $\{z^k\}_{k=0}^{\infty} \subset \mathcal{H}$ is said to be generated by the extended forward-backward splitting(EFBS) scheme if*

$$z^{k+1} \in (H_k + \lambda_k T_2^k)^{-1}(H_k - \lambda_k T_1^k)(z^k), \qquad \forall k \geq 0. \tag{5.2.2}$$

Lemma 5.2.1. *Let $R : \mathcal{H} \to \mathcal{H}$ be any continuous operator. Then*

$$0 \in T^k(z) \quad \text{iff} \quad z \in (R + \lambda T_2^k)^{-1}(R - \lambda T_1^k)(z), \tag{5.2.3}$$

where scalar $\lambda \neq 0$.

Proof. $0 \in T^k(z)$ iff there exist $y_1 \in T_1^k(z)$ and $y_2 \in T_2^k(z)$, such that $y_1 + y_2 = 0$, if and only if, $R(z) - R(z) + \lambda(y_1 + y_2) = 0$ iff $R(z) + \lambda y_2 = R(z) - \lambda y_1$ iff $z \in (R + \lambda T_2^k)^{-1}[R(z) - \lambda y_1]$ iff $z \in (R + \lambda T_2^k)^{-1}(R - \lambda T_1^k)(z)$. ◻

In most cases of our applications, we only need to consider bounded linear operator H_k. Hence, we assume that H_k is a strongly monotone bounded linear operator in the rest of this chapter.

Lemma 5.2.2. *Let R be an invertible bounded linear operator.*

$$\begin{aligned} 0 \in (T_1 + T_2)(z) \quad & \text{iff} \quad 0 \in R(T_1 + T_2)(z) \\ & \text{iff} \quad 0 \in (T_1 R + T_2 R)(R^{-1} z). \end{aligned} \tag{5.2.4}$$

Proof. It is elementary. ◻

Lemma 5.2.3. *Let R be a bounded linear operator and $\bar{R} := \frac{1}{2}(R + R^*)$. Then $\bar{R}$ is symmetric, $R = \bar{R} + (R - \bar{R})$ and*

$$\langle z, (R - \bar{R})z \rangle \equiv 0, \quad \text{for all} \quad z \in \mathcal{H}. \tag{5.2.5}$$

Furthermore, if R is positive definite, or invertible, so is $\bar{R}$.

Proof. For any bounded linear operator R, obviously $\bar{R}$ is symmetric and $R - \bar{R} = \frac{1}{2}(R - R^*)$. Thus, it is easy to see that (5.2.5) is true since $\mathcal{H}$ is a *real* Hilbert space. When R is positive definite or invertible, R^* is also positive definite, or invertible. Thus is $\bar{R}$. ◻

Of course, scheme (5.2.1) is a special case of scheme (5.2.2). Let

$$\begin{aligned} \bar{H}_k = \tfrac{1}{2}(H_k + H_k^*), \quad \bar{T}_1^k = T_1^k - \frac{1}{\lambda_k}(H_k - \bar{H}_k), \\ \bar{T}_2^k = T_2^k + \frac{1}{\lambda_k}(H_k - \bar{H}_k). \end{aligned} \tag{5.2.6}$$

Then, $\bar{H}_k$ is symmetric positive definite, and $\bar{T}_1^k$, $\bar{T}_2^k$ are maximal monotone from Lemma 5.2.3 for each $k \geq 0$. Furthermore, $\bar{T}_1^k + \bar{T}_2^k = T^k$. Therefore, the scheme (5.2.2) is equivalent to the following scheme:

$$z^{k+1} \in (\bar{H}_k + \lambda_k \bar{T}_2^k)^{-1}(\bar{H}_k - \lambda_k \bar{T}_1^k)(z^k). \tag{5.2.7}$$

The following properties relate to keeping the monotonicity of monotone operators under transformations of bounded linear operators.

Proposition 5.2.4. *Suppose T is a monotone operator. If $R : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator, then R^*TR is also monotone.*

Proof. Without loss of generality, suppose T is single-valued. Since T is monotone, we have for any $x', x'' \in \mathcal{H}$

$$\begin{aligned}
&\langle x' - x'', R^*TR(x') - R^*TR(x'') \rangle \\
&= \langle x' - x'', R^*(TRx' - TRx'') \rangle \\
&= \langle R(x' - x''), TR(x') - TR(x'') \rangle \\
&= \langle Rx' - Rx'', T(Rx') - T(Rx'') \rangle \geq 0.
\end{aligned}$$

Hence, R^*TR is monotone. ☐

Proposition 5.2.5. *If $R : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator, with bound $M > 0$, and T^{-1} is strongly monotone, with modulus $\beta > 0$, then $(R^*TR)^{-1}$ is also strongly monotone, with modulus $\frac{\beta}{M^2}$.*

Proof. For any $x', x'' \in \mathcal{H}$, similar to the proof of Proposition 5.2.4,

$$\begin{aligned}
&\langle x' - x'', R^*TR(x') - R^*TR(x'') \rangle \\
&= \langle Rx' - Rx'', T(Rx') - T(Rx'') \rangle \\
&\geq \beta \|T(Rx') - T(Rx'')\|^2 \\
&\geq \frac{\beta}{M^2} \|R\|^2 \|TR(x') - TR(x'')\|^2 \\
&\geq \frac{\beta}{M^2} \|RTR(x') - RTR(x'')\|^2
\end{aligned}$$

Thus, $(R^*TR)^{-1}$ is strongly monotone. ☐

Corollary 5.2.6. *If $\mathcal{H} = \mathbb{R}^n$ and $R \in \mathbb{R}^{n\times n}$, then, when T is monotone, so is R^*TR, and when T^{-1} is strongly monotone, so is $(R^*TR)^{-1}$.*

Proof. This is obvious since R is a bounded linear operator. $\square$

Proposition 5.2.7. *If T is strongly monotone, with modulus $\alpha > 0$, and $R : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator whose inverse operator is still a bounded linear operator R^{-1} with bound M, then R^*TR is strongly monotone with modulus $\frac{\alpha}{M^2}$.*

Proof. Similar to the proof of Proposition 5.2.4, we see

$$\begin{aligned}
&\langle x' - x'', R^*TR(x') - R^*TR(x'')\rangle \\
&= \langle Rx' - Rx'', T(Rx') - T(Rx'')\rangle \\
&\geq \alpha\|Rx' - Rx''\|^2 \\
&= \alpha\|R(x' - x'')\|^2 \\
&\geq \frac{\alpha}{M^2}\|R^{-1}\|^2\|R(x' - x'')\|^2 \\
&\geq \frac{\alpha}{M^2}\|x' - x''\|^2
\end{aligned}$$

$\square$

Corollary 5.2.8. *Suppose T is strongly monotone and $\mathcal{H} = \mathbb{R}^n$. If $R \in \mathbb{R}^{n\times n}$ is nonsingular, then R^*TR is strongly monotone.*

5.3. Some Lemmas for the Convergence of Sequences

In this section, we shall discuss convergence of sequences by the concept of similarity.

Definition 5.3.1. *Suppose E_1 and E_2 are two operators on $\mathcal{H}$. E_1 and E_2 are described as similar if there is an invertible bounded linear operator R such that*

$$E_1 = R^{-1}E_2R \tag{5.3.1}$$

Definition 5.3.2. *Suppose $\{E_k\}_{k=0}^{\infty}$ and $\{G_k\}_{k=0}^{\infty}$ are two operator sets. These two sets of operators are described as uniformly similar if there exists an invertible bounded linear operator R such that*

$$E_k = R^{-1} G_k R, \quad \text{for} \quad k = 0, 1, 2, ... \tag{5.3.2}$$

Lemma 5.3.1. *Suppose E_1 and E_2 are two similar operators and let sequences $\{z^k\}$ and $\{w^k\}$ be from*

$$z^{k+1} \in E_1(z^k), \qquad w^{k+1} \in E_2(w^k), \qquad k = 0, 1, 2, ... \tag{5.3.3}$$

Then, every $\{z^k\}$ strongly converges for some initial z^0, if and only if, every $\{w^k\}$ strongly converges for some initial w^0.

Proof. Assuming (5.3.1) is true, then $E_2 = RE_1R^{-1}$. Thus, we only need to prove one side.
Now, if $\{z^k\}$ strongly converges for z^0, then, let $w^0 = Rz^0$ and $u^k = R^{-1}w^k$ for $k = 0, 1, 2, ...$ and we have

$$u^{k+1} \in E_1(u^k), \quad \text{with} \quad u^0 = R^{-1}w^0 = z^0.$$

So, $\{u^k\}$ converges strongly. However, R is a bounded operator. Therefore, $\{w^k\} = \{Ru^k\}$ strongly converges for $w^0 = Rz^0$. □

Lemma 5.3.2. *Suppose $\{E_k\}_{k=0}^{\infty}$ and $\{G_k\}_{k=0}^{\infty}$ are two uniformly similar operator sets. Let sequences $\{z^k\}$ and $\{w^k\}$ be from*

$$z^{k+1} \in E_k(z^k), \qquad w^{k+1} \in G_k(w^k), \qquad k = 0, 1, 2, ... \tag{5.3.4}$$

Then, every $\{z^k\}$ strongly converges for some initial z^0, if and only if, every $\{w^k\}$ strongly converges for some initial w^0.

Proof. Similar to the proof of Lemma 5.3.1. □

Now consider the following iteration:

$$w^{k+1} \in S_k(w^k). \tag{5.3.5}$$

Where $S_k : \mathcal{H} \rightrightarrows \mathcal{H}$.

From Theorem 4.3.3, if $\{S_k\}$ is uniformly contractive and $w^* = S_k(w^*)$ for every $k \geq 0$, then $\{w^k\}$ from (5.3.5) Q-linearly converges to w^*.

Definition 5.3.3. *For an operator $T : \mathcal{H} \rightrightarrows \mathcal{H}$, define*

$$\mathrm{Fix}(T) := \{z : z \in T(z)\}. \tag{5.3.6}$$

Now consider the following iteration scheme:

$$z^{k+1} \in R_k^{-1} S_k R_k(z^k), \qquad k = 0, 1, 2, \ldots. \tag{5.3.7}$$

Where R_k is an invertible bounded linear operator for each k.

Lemma 5.3.3. *If $\mathrm{Fix}(S_k) = \mathcal{F}_k$, then $\mathrm{Fix}(R_k S_k R_k^{-1}) = R_k \mathcal{F}_k$. If $S_k \to S'$ and $R_k \to R'$ as $k \to \infty$, then $\mathcal{F}_k \to \mathrm{Fix}(S')$ and $\mathrm{Fix}(R_k S_k R_k^{-1}) \to R' \,\mathrm{Fix}(S')$.*

Proof. $x \in S_k(x) \iff R_k x \in R_k S_k(x) = R_k S_k R_k^{-1}(R_k x)$. Thus, $\mathrm{Fix}(R_k S_k R_k^{-1}) = R_k \mathcal{F}_k$. The second conclusion is elementary. □

Theorem 5.3.4. *Suppose $\{S_k\}$ is uniformly contractive with constant c and $R_k \to R'$, $R_k^{-1} \to (R')^{-1}$ with R' invertible. If $\mathrm{Fix}(R_k^{-1} S_k R_k) \equiv \mathrm{zer}(T)$, then $\{z^k\}$ from (5.3.7) strongly converges to the zero of T. Furthermore, convergence rate will be Q-linear with global ratio c, when $R_k \equiv R'$.*

Proof. Let $w^k = R_k z^k$. Then

$$w^{k+1} = R_{k+1} R_k^{-1} S_k(w^k), \qquad R_k z^* \in \mathrm{Fix}(S_k) \quad \text{with} \quad z^* \in \mathrm{zer}(T).$$

Hence

$$w^{k+1} - R_{k+1} z^* = R_{k+1} R_k^{-1}(S_k(w_k) - S_k(R_k z^*)).$$

Since $R_{k+1} R_k^{-1} \to I$ as $k \to \infty$, there exists a $N \geq 0$ for any $c < \eta < 1$, such that $c\|R_{k+1} R_k^{-1}\| \leq \eta$ when $k \geq N$. So, when $k > N$,

$$\begin{aligned} \|w^{k+1} - R_{k+1} z^*\| &\leq \eta \|w^k - R_k z^*\| \\ &\leq \eta^{k+1-N} \|w^N - R_N z^*\|. \end{aligned}$$

This implies $w^k \to R' z^*$ as $k \to \infty$. Therefore, $z^k = R_k^{-1} w^k$ approaches $(R')^{-1} R' z^* = z^*$.

When $R_k \equiv R'$,

$$\|w^{k+1} - R' z^*\| \leq c \|w^k - R' z^*\| \quad \text{for all} \quad k \geq 0.$$

This implies the Q-linear convergence. □

Lemma 5.3.5. *Suppose $\{R_k\}$ and R' are invertible bounded linear operators. When $R_k \to R'$ as $k \to \infty$,and if $\{\|R_k^{-1}\|\}$ is bounded, or if R' is strongly monotone, or if $\mathcal{H} = \mathbb{R}^n$, or if R' is symmetric and monotone, then $R_k^{-1} \to (R')^{-1}$.*

Proof. If $\mathcal{H} = \mathbb{R}^n$, the conclusion is elementary from Algebra.
Now, since

$$\|R_k^{-1} - (R')^{-1}\| \leq \|R_k^{-1}\|\|(R')^{-1}\|\|R' - R_k\|,$$

so, if $\{\|R_k^{-1}\|\}$ is a bounded set, the conclusion is true. If R' is strongly monotone, let its modulus be α. Then there is a nonnegative integer J such that R_k is strongly monotone with modulus $\frac{1}{2}\alpha$ for all $k \geq J$. Hence, for $k \geq J$ and any x with norm 1, there exists a unique z such that $R_k^{-1}x = R_k z$. Thus,

$$\|R_k^{-1}x\| \leq \|R_k\|\|z\| \leq \frac{4}{\alpha^2}\|R_k\|,$$

because $1 = \|x\| = \|R_k^2 z\| \geq \frac{\alpha^2}{4}\|z\|$. $R_k \to R'$ implies that $\{\|R_k^{-1}\|\}$ is a bounded set.
If R' is symmetric and monotone, then it is strongly monotone from Lemma 6.9 of [17]. □

Before we prove the next two convergence results, we will introduce an important lemma which is found in Chapter 2 of the Polyak's book (1987).

Lemma 5.3.6. *Let*

$$u_{k+1} \leq q_k u_k + \varepsilon_k, \qquad 0 \leq q_k < 1, \quad \varepsilon_k \geq 0,$$

$$\sum_{k=0}^{\infty}(1 - q_k) = \infty, \qquad \frac{\varepsilon_k}{1 - q_k} \to 0.$$

Then, $\lim_{k\to\infty} u_k \leq 0$. In particular, if $u_k \geq 0$, then, $\lim_{k\to\infty} u_k = 0$. □

Theorem 5.3.7. *Suppose $R_k \equiv R$ given R invertible and $\{S_k\}$ is a uniformly contractive sequence with constant c in (5.3.7). Let $\{z^k\}$ be a sequence generated from (5.3.7). If*

$$\lim_{k\to\infty} \bar{w}_k = w^* \qquad \textit{given} \quad \bar{w}_k = S_k(\bar{w}_k) \quad k = 0, 1, \ldots, \tag{5.3.8}$$

then, $z^k \to R^{-1}w^*$. *Moreover, if there exists a* $\eta < 1$ *such that*

$$\|\bar{w}_k - w^*\| \le \eta^k C_0 \qquad \text{for any} \quad k \ge 0, \tag{5.3.9}$$

where C_0 *is a constant, then the convergence rate is linear.*

Proof. Let $w^k = Rz^k$. Then, (5.3.7) implies $w^{k+1} = S_k(w^k)$. Hence, $\bar{w}_k = S_k(\bar{w}_k)$ for $k = 0, 1, \ldots$, it follows that

$$\begin{aligned} \|w^{k+1} - w^*\| &= \|S_k(w^k) - S_k(w^*) + S_k(w^*) - S_k(\bar{w}_k) + \bar{w}_k - w^*\| \\ &\le c\|w^k - w^*\| + (1+c)\|\bar{w}_k - w^*\|. \end{aligned} \tag{5.3.10}$$

(5.3.8) and Lemma 5.3.6 imply that

$$\lim_{k\to\infty} w^k \;=\; \lim_{k\to\infty} \bar{w}_k \;=\; w^*.$$

Therefore,

$$\lim_{k\to\infty} z^k \;=\; \lim_{k\to\infty} R^{-1}w^k \;=\; R^{-1}w^*.$$

If (5.3.9) is satisfied, (5.3.10) will imply

$$\|w^k - w^*\| \le c^k\|w^0 - w^*\| + (1+c)\sum_{i=0}^{k-1} c^{k-i-1}\eta^i C_0.$$

Let

$$\tau = \max\{\sqrt{c}, \sqrt{\eta}\} < 1.$$

Then, if

$$\bar{C} = \sup_{k>0}\left\{(1+c)\tau^{k-2}kC_0 + \tau^k\|w^0 - w^*\|\right\},$$

we have $\bar{C} < \infty$ and

$$\|w^k - w^*\| \le \tau^k \bar{C} \qquad \text{for} \quad k > 0.$$

Thus, the convergence rate is linear. $\square$

Theorem 5.3.8. *Suppose $R_k \equiv R$ given R invertible and S_k is contractive with constant c_k in (5.3.7) for $k = 0, 1, \cdots$. Let $\{z^k\}$ be a sequence generated from (5.3.7). If*

$$\sum_{k=0}^{\infty}(1-c_k) = \infty, \qquad \lim_{k\to\infty}\frac{\|\bar{w}_k - w^*\|}{1-c_k} = 0 \tag{5.3.11}$$

with $\bar{w}_k = S_k(\bar{w}_k)$ for $k = 0, 1, \cdots$, then $z^k \to R^{-1}w^*$.

Proof. Let $w^k = Rz^k$. Then (5.3.7) implies $w^{k+1} = S_k(w^k)$. Hence, $\bar{w}_k = S_k(\bar{w}_k)$ for $k = 0, 1, \ldots$, it follows that

$$\begin{aligned}\|w^{k+1} - w^*\| &= \|S_k(w^k) - S_k(w^*) + S_k(w^*) - S_k(\bar{w}_k) + \bar{w}_k - w^*\| \\ &\le c_k\|w^k - w^*\| + (1+c_k)\|\bar{w}_k - w^*\|.\end{aligned}$$

(5.3.11) and Lemma 5.3.6 imply that

$$\lim_{k\to\infty} w^k = w^*.$$

Therefore,

$$\lim_{k\to\infty} z^k = \lim_{k\to\infty} R^{-1}w^k = R^{-1}w^*.$$

□

5.4. Global Convergence and Rates of Extended FBS Methods

Now, let us consider the convergence of extended FBS methods.

Assume that S is a symmetric positive definite bounded linear operator with bound N_1. Hence, S is invertible. Let N be a bound of S^{-1}.

So, we have the convergence theorem as follows.

Theorem 5.4.1. *Suppose $T = T_1 + T_2$ with T_1, T_2 maximal monotone and $\{z^k\}$ is generated from (5.2.2) with $H_k \equiv S^2$ and $T_1^k = T_1$, $T_2^k = T_2$. Then*

(a) $\{z^k\}$ weakly converges to a zero point of T, if $T^{-1}(0) \neq \emptyset$, T_1^{-1} is strongly monotone with modulus β and

$$0 < \inf_{k\ge 0}\{\lambda_k\} \le \sup_{k\ge 0}\{\lambda_k\} < \frac{2\beta}{N^2}, \tag{5.4.1}$$

(b) $\{z^k\}$ *will strongly converge to the unique zero point of* T *with Q-linear rate if one of the following is satisfied:*

(b.1) T_1 *is Lipschitz continuous with constant* M*,* T_2 *is strongly monotone with modulus* α *and*

$$0 < \inf_{k\geq 0}\{\lambda_k\} \leq \sup_{k\geq 0}\{\lambda_k\} < \frac{2\alpha N_1^{-2}}{\max\{0, M^2N^4 - N_1^{-4}\alpha^2\}}, \tag{5.4.2}$$

(b.2) T_1 *is Lipschitz continuous with constant* M*,* T_1 *and* T_2 *are strongly monotone with moduli* α_1 *and* α_2 *respectively and*

$$0 < \inf_{k\geq 0}\{\lambda_k\} \leq \sup_{k\geq 0}\{\lambda_k\} < \frac{2(\alpha_1+\alpha_2)N_1^{-2}}{\max\{0, M^2N^4 - N_1^{-4}\alpha_2^2\}}. \tag{5.4.3}$$

(b.3) T_1 *is Lipschitz continuous with constant* M *and strongly monotone with modulus* α *and*

$$0 < \inf_{k\geq 0}\{\lambda_k\} \leq \sup_{k\geq 0}\{\lambda_k\} < \frac{2\alpha}{(N_1MN^2)^2}, \tag{5.4.4}$$

Proof. Let $w^k = Sz^k$. From Lemma 5.3.2 and the BFBS Theorem in Section 5.2, we only need to prove that the $\{w^k\}$ weakly converges to a zero point of $S^{-1}TS^{-1}$ under the conditions in Part (a) and Q-linearly converges to the unique zero point of $S^{-1}TS^{-1}$ under one of the conditions (b.1), (b.2), (b.3). Where $\{w^k\}$ satisfies

$$w^{k+1} \in (I + \lambda_k S^{-1}T_2S^{-1})^{-1}(I - \lambda_k S^{-1}T_1S^{-1})(w^k). \tag{5.4.5}$$

When T_1 is Lipschitz continuous with constant M, $S^{-1}T_1S^{-1}$ is also Lipschitz continuous with constant MN^2. Therefore, $\{w^k\}$ does converge as above for any initial point w^0 from Proposition 5.2.5, Proposition 5.2.7, and the BFBS Theorem in Section 5.2. □

For the next theorem, the following concepts are required.

Definition 5.4.1. *Let* $G_k : \mathcal{H} \rightrightarrows \mathcal{H}$ *for* $k = 0, 1, 2, \ldots$ *be a sequence of operators. We describe* $\{G_k\}$ *as uniformly Lipschitz continuous if there is a constant* $N \geq 0$ *such that for any* $k \geq 0$

$$\|y' - y''\| \leq N\|x' - x''\| \forall x', x'' \in \mathcal{H}, \quad y' \in G_k(x'), y'' \in G_k(x''). \tag{5.4.6}$$

Definition 5.4.2. *We describe $\{G_k\}$ as uniformly strongly monotone with modulus α, if $\alpha > 0$, and for any $k \geq 0$ we have for any $x', x'' \in \mathcal{H}$ that*

$$\langle x' - x'', y' - y'' \rangle \geq \alpha \|x' - x''\|^2, \quad \forall \quad y' \in G_k(x'), \quad y'' \in G_k(x''). \tag{5.4.7}$$

Theorem 5.4.2. *Suppose* $\mathrm{zer}(T^k) = \mathrm{zer}(T) \neq \emptyset$ *and* $H_k \equiv S^2$. *If* $\{z^k\}$ *is generated from (5.2.2), then*

(a) $\{z^k\}$ *will weakly converge to a zero of* T *when* $\{S(T_1^k)^{-1}S\}$ *is uniformly strongly monotone with modulus* β, *(2.10.9) is satisfied, and* $T^k \to T$ *as* $k \to \infty$.

(b) $\{z^k\}$ *will Q-linearly converge to the unique zero point of* T *if one of the following conditions is true:*

(b.1) $\{S^{-1}T_1^kS^{-1}\}$ *is uniformly Lipschitz continuous with constant* M, $\{\lambda_k\}$ *satisfies (4.2.3), and* $\{S^{-1}T_2^kS^{-1}\}$ *is uniformly strongly monotone with modulus* α.

(b.2) $\{S^{-1}T_1^kS^{-1}\}$ *is uniformly Lipschitz continuous with constant* M, $\{\lambda_k\}$ *satisfies (4.2.11),* $\{S^{-1}T_1^kS^{-1}\}$ *and* $\{S^{-1}T_2^kS^{-1}\}$ *are uniformly strongly monotone with moduli* α_1 *and* α_2 *respectively.*

(b.3) $\{S^{-1}T_1^kS^{-1}\}$ *is uniformly Lipschitz continuous with constant* M *and uniformly strongly monotone with modulus* α *and (4.3.1) is satisfied.*

Proof. Let $w^k = Sz^k$. From Lemma 5.3.2, we only need to prove that $\{w^k\}$ has the same convergence properties as $\{z^k\}$ under the conditions in the theorem. Here $\{w^k\}$ satisfies

$$w^{k+1} = (I + \lambda_k S^{-1}T_2^kS^{-1})^{-1}(I - \lambda_k S^{-1}T_1^kS^{-1})(w^k). \tag{5.4.8}$$

<u>Proof of (a) :</u>

Let $\bar{T}_1^k = S^{-1}T_1^kS^{-1}$, $\bar{T}_2^k = S^{-1}T_2^kS^{-1}$ and $y^{k+1} \in \bar{T}_2^k(w^{k+1})$ be such that

$$w^{k+1} + \lambda_k y^{k+1} = w^k - \lambda_k \bar{T}_1^k(w^k). \tag{5.4.9}$$

Note that T_1^k or $\bar{T}_1^k$ is single-valued under any one of conditions in the theorem. Let $z^* \in \mathrm{zer}(T)$ and $y_k^* = -\bar{T}_1^k(w^*)$ with $w^* = Sz^*$. Then, $y_k^* \in \bar{T}_2^k(w^*)$. Hence

$$w^{k+1} - w^* + \lambda_k(y^{k+1} - y_k^*) = w^k - w^* - \lambda_k(\bar{T}_1^k(w^k) - \bar{T}_1^k(w^*)).$$

So, if $\eta = \min\{\inf\{\lambda_k\}, 2\beta - \sup\{\lambda_k\}\}$, then $\eta > 0$, and

$$\|w^{k+1} - w^*\|^2 + \eta^2\|y^{k+1} - y_k^*\|^2 \le \|w^k - w^*\|^2 - \eta^2\|\bar{T}_1^k(w^k) - \bar{T}_1^k(w^*)\|^2$$

Adding the above over $k = 0, 1, \ldots, m-1$ gives

$$\begin{aligned} &\|w^m - w^*\|^2 + \eta^2 \sum_{k=0}^{m-1} \|y^{k+1} - y_k^*\|^2 \\ &\le \|w^0 - w^*\|^2 - \eta^2 \sum_{k=0}^{m-1} \|\bar{T}_1^k(w^k) - \bar{T}_1^k(w^*)\|^2. \end{aligned} \tag{5.4.10}$$

(5.4.10) implies that $\{w^k\}$ is bounded and the two positive term series are convergent as $m \to \infty$. Thus

$$\lim_{k\to\infty} \bar{T}_1^k(w^k) - \bar{T}_1^k(w^*) = 0, \qquad \lim_{k\to\infty} y^{k+1} - y_k^* = 0 \tag{5.4.11}$$

There is a weak cluster point w^∞ and a subsequence $\{w^{k(j)}\}$ of $\{w^k\}$ such that

$$w^{k(j)} \to w^\infty \quad \text{weakly as} \quad j \to \infty. \tag{5.4.12}$$

Now, (2.10.9), (5.4.9), (5.4.11), and Proposition 2.1.5 imply that

$$\begin{aligned} &\|\bar{T}_1^k(w^{k+1}) - \bar{T}_1^k(w^k)\| \le \beta^{-1}\|w^{k+1} - w^k\| \\ &= \beta^{-1}\lambda_k\|y^{k+1} + \bar{T}_1^k(w^k)\| \to 0 \quad \text{as} \quad k \to \infty. \end{aligned} \tag{5.4.13}$$

$T^k \to T$ implies that $\bar{T}^k = \bar{T}_1^k + \bar{T}_2^k \to \bar{T} = S^{-1}TS^{-1}$ pointwisely. Hence for any $w \in \mathcal{H}$ and $x \in \bar{T}(w)$, there is a sequence $\{x^k\}$ with $x^k \in \bar{T}^k(w)$ such that $x^k \to x$. Thus, Lemma 4.1.8, (5.4.11), (5.4.12) and (5.4.13) give the following:

$$\begin{aligned} 0 &\le \lim_{j\to\infty} \langle w^{k(j)} - w, y^{k(j)} + \bar{T}_1^{k(j)-1}(w^{k(j)}) - x^{k(j)-1}\rangle \\ &= \langle w^\infty - w, -x\rangle \end{aligned} \tag{5.4.14}$$

for any $x \in \bar{T}(w)$ and $w \in \mathcal{H}$. From the maximal monotonicity of T and, hence, $\bar{T}$, (5.4.14) tells us that $0 \in \bar{T}(w^\infty)$. Therefore, $z^\infty = S^{-1}w^\infty$ is a zero point of T. However, (2.10.9), (5.4.11), and

$$w^{k+1} = w^k - \lambda_k[y^{k+1} + \bar{T}_1^k(w^k)]$$

imply that $\{w^k\}$ actually weakly converges to w^∞. That is, $\{z^k\}$ weakly converges to z^∞ with $0 \in T(z^\infty)$.

Proof of (b) :

Let $S_k = (I + \lambda_k \bar{T}_2^k)^{-1}(I - \lambda_k \bar{T}_1^k)$ in (5.3.5) and $R_k \equiv S$ in (5.3.7). Then, if the above $\{S_k\}$ is uniformly contractive when one of the conditions in Part (b) is satisfied, the conditions of Theorem 5.3.4 are satisfied. Hence the conclusion is true. However, the uniform contractivity of $\{S_k\}$ can be proved by directly using the proof of Theorem 4.3.4, or Theorem 4.3.5, or Theorem 4.3.6 and just changing G_k, T_1 and T_2 to S_k, $\bar{T}_1^k$ and $\bar{T}_2^k$ here. $\square$

Note that Theorem 5.4.1 is a special case of Theorem 5.4.2. Now, since (5.2.2) is a special case of (5.3.7), then, when $\mathrm{zer}(T^k) \equiv \mathrm{zer}(T)$, Lemma 5.3.4 can be again applied to $\{z^k\}$ from (5.2.2) or, equivalently, (5.2.7) and implies a more general theorem similar to Part (b) of Theorem 5.4.2.

Theorem 5.4.3. *Suppose $\{z^k\}$ comes from (5.2.7) and $z^* \in \mathrm{zer}(T) \bigcap_k \mathrm{zer}(T^k)$. If $\{\bar{H}_k\}$ converges to an invertible bounded linear operator H', then $\{z^k\}$ strongly converges to z^* under one of the following conditions:*

(a) *$\{\bar{H}_k^{-\frac{1}{2}} T_2^k \bar{H}_k^{-\frac{1}{2}}\}$ is uniformly strongly monotone with modulus α, $\{\lambda_k\}$ satisfies (4.2.3), and $\{\bar{H}_k^{-\frac{1}{2}} T_1^k \bar{H}_k^{-\frac{1}{2}}\}$ is uniformly Lipschitz continuous with constant M.*

(b) *$\{\bar{H}_k^{-\frac{1}{2}} T_1^k \bar{H}_k^{-\frac{1}{2}}\}$ is uniformly Lipschitz continuous with constant M, $\{\lambda_k\}$ satisfies (4.2.11), $\{\bar{H}_k^{-\frac{1}{2}} T_1^k \bar{H}_k^{-\frac{1}{2}}\}$ and $\{\bar{H}_k^{-\frac{1}{2}} T_2^k \bar{H}_k^{-\frac{1}{2}}\}$ are uniformly strongly monotone with moduli α_1 and α_2 respectively.*

(c) *$\{\bar{H}_k^{-\frac{1}{2}} T_1^k \bar{H}_k^{-\frac{1}{2}}\}$ is uniformly Lipschitz continuous with constant M and uniformly strongly monotone with modulus α, and $\{\lambda_k\}$ satisfies (4.3.1).*

Proof. These are special cases of Theorem 5.3.4 similar to the proof of Part (b) in Theorem 5.4.2. $\square$

Theorem 5.4.4. *Suppose $\{z^k\}$ is generated from (5.2.2), for any given z^0, and $H_k = S^2$ with S symmetric positive definite. If $\bar{z}_k \in \mathrm{zer}(T^k)$ for $k = 0, 1, \ldots$ and*

$$\lim_{k\to\infty} \bar{z}_k = z^* \in \mathrm{zer}(T), \tag{5.4.15}$$

then, $\{z^k\}$ *strongly converges to* z^* *under one of the following three conditions:*

(a) $\{S^{-1}T_2^k S^{-1}\}$ *is uniformly strongly monotone with modulus* α, $\{S^{-1}T_1^k S^{-1}\}$ *is uniformly Lipschitz continuous with constant* M, *and* $\{\lambda_k\}$ *satisfies (4.2.3).*

(b) $\{S^{-1}T_1^k S^{-1}\}$ *is uniformly Lipschitz continuous with constant* M, $\{\lambda_k\}$ *satisfies (4.2.11),* $\{S^{-1}T_1^k S^{-1}\}$ *and* $\{S^{-1}T_2^k S^{-1}\}$ *are uniformly strongly monotone with moduli* α_1 *and* α_2, *respectively.*

(c) $\{S^{-1}T_1^k S^{-1}\}$ *is uniformly Lipschitz continuous with constant* M *and uniformly strongly monotone with modulus* α, *and* $\{\lambda_k\}$ *satisfies (4.3.1).*

Moreover, the convergence is linear, if there is a $\eta < 1$ *and a constant* $\bar{C}$, *such that*

$$\|\bar{z}_k - z^*\| \leq \eta^k \bar{C} \qquad \text{for all} \quad k \geq 0. \tag{5.4.16}$$

Proof. These are special cases of Theorem 5.3.7 similar to the proof of Part (b) in Theorem 5.4.2. □

Theorem 5.4.5. *Let* $\{z^k\}$ *be generated from (5.2.2) for any given* z^0 *and* $H_k = S^2$ *with* S *symmetric positive definite. Suppose that* $\{S(T_1^k)^{-1}S\}$ *is uniformly strongly monotone with modulus* β, *(2.10.9) is satisfied and* $T^k \to T$ *when* $k \to \infty$. *If* $\bar{z}_k \in \mathrm{zer}(T^k)$ *for* $k = 0, 1, \cdots$ *and*

$$\sum_{k=0}^{\infty} \|\bar{z}_{k+1} - \bar{z}_k\| < \infty, \tag{5.4.17}$$

then

(a) $\mathrm{zer}(T) \neq \emptyset$ *and* $\{z^k\}$ *will weakly converge to a zero of* T;

(b)

$$\lim_{k\to\infty} \|z^{k+1} - z^k\| = 0, \qquad \sum_{k=0}^{\infty} \|z^{k+1} - z^k\|^2 < \infty. \tag{5.4.18}$$

Proof. Let $w^k = Sz^k$. From Lemma 5.3.2, we only need to prove that $\{w^k\}$ has the same convergent properties as $\{z^k\}$ under the conditions in the theorem. Where $\{w^k\}$ satisfies (5.4.8).

Let $\bar{T}_1^k = S^{-1}T_1^k S^{-1}$, $\bar{T}_2^k = S^{-1}T_2^k S^{-1}$, and $y^{k+1} \in \bar{T}_2^k(w^{k+1})$ be such that

$$w^{k+1} + \lambda_k y^{k+1} = w^k - \lambda_k \bar{T}_1^k(w^k). \tag{5.4.19}$$

Note that T_1^k or $\bar{T}_1^k$ is single-valued under the given conditions in the theorem. Let $w_k^* = S\bar{z}_k$ and $y_k^* = -\bar{T}_1^k(w_k^*)$. Then, $y_k^* \in \bar{T}_2^k(w_k^*)$. Hence

$$w^{k+1} - w_k^* + \lambda_k(y^{k+1} - y_k^*) = w^k - w_k^* - \lambda_k(\bar{T}_1^k(w^k) - \bar{T}_1^k(w_k^*)).$$

So, if $\eta = \min\{\inf\{\lambda_k\}, 2\beta - \sup\{\lambda_k\}\}$, then, $\eta > 0$ and

$$\begin{aligned} &\|w^{k+1} - w_k^*\|^2 + \eta^2\|y^{k+1} - y_k^*\|^2 \\ \leq &\|w^k - w_k^*\|^2 - \eta^2\|\bar{T}_1^k(w^k) - \bar{T}_1^k(w_k^*)\|^2. \end{aligned} \tag{5.4.20}$$

Thus,

$$\begin{aligned} \|w^{k+1} - w_k^*\| &\leq \|w^k - w_k^*\| \\ &\leq \|w^k - w_{k-1}^*\| + \|w_{k-1}^* - w_k^*\| \\ &\leq \|w^1 - w_0^*\| + \sum_{i=1}^{k} \|w_{i-1}^* - w_i^*\|. \end{aligned}$$

Since $\{\bar{z}_k\}$ is bounded from (5.4.17), S is a bounded operator and (5.4.17) is true. Thus the above inequalities imply that $\{w^k\}$ and $\{w^{k+1} - w_k^*\}$ are bounded sequences. Hence, there exists a weak cluster point w^∞ and a subsequence $\{w^{k(j)}\}$ of $\{w^k\}$, such that

$$w^{k(j)} \to w^\infty \quad \text{weakly as} \quad j \to \infty. \tag{5.4.21}$$

Now, adding (5.4.20) over $k = 1, 2, \ldots, m$ gives

$$\begin{aligned} &\|w^{m+1} - w_m^*\|^2 + \sum_{k=1}^{m} \left[\eta^2\|y^{k+1} - y_k^*\|^2 + 2\langle w_{k-1}^* - w^k, w_{k-1}^* - w_k^*\rangle\right] \\ \leq\ &\|w^1 - w_0^*\|^2 + \sum_{k=1}^{m} \|w_{k-1}^* - w_k^*\|^2 - \eta^2 \sum_{k=1}^{m} \|\bar{T}_1^k(w^k) - \bar{T}_1^k(w^*)\|^2. \end{aligned}$$

The boundedness of $\{w^{k+1} - w_k^*\}$, (5.4.17), and the above imply that the two positive term series are convergent as $m \to \infty$. That is,

$$\sum_{k=1}^{\infty} \|y^{k+1} - y_k^*\|^2 < \infty, \qquad \sum_{k=1}^{\infty} \|\bar{T}_1^k(w^k) - \bar{T}_1^k(w_k^*)\|^2 < \infty.$$

Thus

$$\lim_{k\to\infty} \bar{T}_1^k(w^k) - \bar{T}_1^k(w_k^*) = 0, \qquad \lim_{k\to\infty} y^{k+1} - y_k^* = 0. \tag{5.4.22}$$

Now, (2.10.9), (5.4.19), (5.4.22), and Proposition 2.1.5 imply that

$$\begin{aligned} \|\bar{T}_1^k(w^{k+1}) - \bar{T}_1^k(w^k)\| &\le \beta^{-1}\|w^{k+1} - w^k\| \\ &= \beta^{-1}\lambda_k\|y^{k+1} + \bar{T}_1^k(w^k)\| \to 0 \quad \text{as} \quad k \to \infty. \end{aligned} \tag{5.4.23}$$

$T^k \to T$ implies that $\bar{T}^k = \bar{T}_1^k + \bar{T}_2^k \to \bar{T} = S^{-1}TS^{-1}$ pointwisely. Hence, for any $w \in \mathcal{H}$ and $x \in \bar{T}(w)$, there is a sequence $\{x^k\}$ with $x^k \in \bar{T}^k(w)$, such that $x^k \to x$. Thus, (5.4.21), (5.4.22), (5.4.23), and Lemma 4.1.8 give the following:

$$0 \le \lim_{j\to\infty} \langle w^{k(j)} - w, y^{k(j)} + \bar{T}_1^{k(j)-1}(w^{k(j)}) - x^{k(j)-1}\rangle = \langle w^\infty - w, -x\rangle \tag{5.4.24}$$

for any $x \in \bar{T}(w)$ and $w \in \mathcal{H}$. From the maximal monotonicity of T and, hence, $\bar{T}$, (5.4.24) tells us that $0 \in \bar{T}(w^\infty)$. Therefore, $z^\infty = S^{-1}w^\infty$ is a zero point of T. However, (2.10.9), (5.4.22) and

$$w^{k+1} = w^k - \lambda_k[y^{k+1} + \bar{T}_1^k(w^k)]$$

imply that $\{w^k\}$ actually weakly converges to w^∞. That is, $\{z^k\}$ weakly converges to z^∞ with $0 \in T(z^\infty)$. The proof of Part (a) is, thus, complete.

For Part (b), (5.4.19) and (5.4.22) imply that $\|w^{k+1} - w^k\| \to 0$, and, hence, $\|z^{k+1} - z^k\| \to 0$ as $k \to \infty$. Now

$$\begin{aligned} \sum_{k=0}^{\infty} \|w^{k+1} - w^k\|^2 &= \sum_{k=0}^{\infty} \lambda_k^2\|y^{k+1} + \bar{T}_1^k(w^k)\|^2 \\ &\le \sum_{k=0}^{\infty} 2\lambda_k^2\left[\|y^{k+1} - y_k^*\|^2 + \|\bar{T}_1^k(w^k) - \bar{T}_1^k(w_k^*)\|^2\right] < \infty \end{aligned}$$

since $y_k^* + \bar{T}_1^k(w_k^*) = 0$ and $0 < \lambda_k < 2\beta$. Hence, $w^k = Sz^k$ implies (5.4.18). □

For the fundamental problem $(\mathcal{P}_0)$ with T not strongly monotone, Theorem 5.3.8 suggests some ideas to solve for $(\mathcal{P}_0)$. If we choose $\{T^k\}$, $\{\lambda_k\}$, and $H = S^2$, such that, the extended forward-backward operators

$$E_k = (I + \lambda_k S^{-1}T_2^k S^{-1})^{-1}(I - \lambda_k S^{-1}T_1^k S^{-1}) \tag{5.4.25}$$

satisfy the conditions in Theorem 5.3.8, then the extended FBS method will converge to a zero point of T if $S^{-1}w^* \in \operatorname{zer}(T)$ in (5.3.11). Please note the following example:

Theorem 5.4.6. *Suppose $\{z^k\}$ is generated from (5.2.2), for any given z^0, and $H_k = S^2$ with S symmetric positive definite. If $\bar{z}_k \in \mathrm{zer}(T^k)$ for $k = 0, 1, \dots$ and $\mathrm{zer}(T) \neq \emptyset$, then $\{z^k\}$ strongly converges to a zero point z^* of T under one of the following three conditions:*

(a) $S^{-1}T_1^k S^{-1}$ is Lipschitz continuous, with constant M_k, and strongly monotone, with modulus α_k,

$$0 < \lambda_k < 2\alpha_k M_k^{-2}, \qquad c_k = 1 - 2\alpha_k\lambda_k + \lambda_k^2 M_k^2, \tag{5.4.26}$$

and there is a $z^* \in \mathrm{zer}(T)$ such that

$$\sum_{k=0}^{\infty}(1 - c_k) = \infty, \qquad \lim_{k\to\infty} \frac{\|\bar{z}_k - z^*\|}{1 - c_k} = 0. \tag{5.4.27}$$

(b) $S^{-1}T_1^k S^{-1}$ is Lipschitz continuous, with constant M_k, $S^{-1}T_2^k S^{-1}$ is strongly monotone, with modulus α_k,

$$0 < \lambda_k < \begin{cases} \frac{2\alpha_k}{M_k^2 - \alpha_k^2}, & \text{for } M_k > \alpha_k; \\ \infty, & \text{for } M_k \leq \alpha_k \end{cases} \tag{5.4.28}$$

and there is a $z^* \in \mathrm{zer}(T)$ such that (5.4.27) is true. Where

$$c_k = \frac{1 + \lambda_k^2 M_k^2}{(1 + \lambda_k\alpha_k)^2} \tag{5.4.29}$$

(c) $S^{-1}T_1^k S^{-1}$ is Lipschitz continuous, with constant M_k, $S^{-1}T_1^k S^{-1}$ and $S^{-1}T_2^k S^{-1}$ are strongly monotone, with moduli α_k' and α_k'', respectively,

$$0 < \lambda_k < \begin{cases} \frac{2(\alpha_k' + \alpha_k'')}{M_k^2 - (\alpha_k'')^2}, & \text{for } M_k > \alpha_k''; \\ \infty, & \text{for } M_k \leq \alpha_k'' \end{cases} \tag{5.4.30}$$

and there is a $z^* \in \mathrm{zer}(T)$ such that (5.4.27) is true. Where

$$c_k = \frac{1 - 2\alpha_k'\lambda_k + \lambda_k^2 M_k^2}{(1 + \lambda_k\alpha_k'')^2}. \tag{5.4.31}$$

Proof. Similar to the proofs of Theorem 4.3.4, Theorem 4.3.5, and Theorem 4.3.6, we see the operator E_k, given by (5.4.25), is strongly monotone with modulus c_k for every $k \geq 0$, under one of the conditions. However, if $\bar{w}_k = S\bar{z}_k$ and $w^* = Sz^*$, then it follows that $\bar{w}_k = E_k(\bar{w}_k)$ and (5.3.11) is satisfied under one of the conditions. Therefore, (5.4.8) and Theorem 5.3.8 imply our conclusion with $S_k \equiv E_k$ and $R = S$. □

5.5. The Local Convergence and Rate of the Extended FBS Method

Now, let us consider the local convergence of the extended FBS method by applying the concepts in Section 4.7 and the previous results.

Definition 5.5.1. *Let $\{T_i \mid i \in U\}$ be a set of operators with an index set U.*

(a) We describe $\{T_i \mid i \in U\}$ as (locally) uniformly strongly monotone at z^*, if every T_i is strongly monotone at z^*, with a same modulus α and a same constant δ for all $i \in U$, under Definition 4.7.1.

(b) We describe $\{T_i \mid i \in U\}$ as (locally) uniformly Lipschitz continuous at z^*, if every T_i is Lipschitz continuous at z^*, with a same Lipschitz constant M and a same η, under Definition 4.7.2.

Definition 5.5.2. *Let $\{T_i \mid i \in U\}$ be a set of operators with an index set U.*

(a) $\{T_i \mid i \in U\}$ are (locally) consistently contractive with constant c (and bound κ) at $\{z_i \mid i \in U\}$, if every T_i is locally contractive at z_i, with the same constant $c < 1$, and with the same $\kappa > 0$ under Definition 4.7.3.

(b) $\{T_i \mid i \in U\}$ are (locally) consistently strongly monotone, with modulus α (and bound δ) at $\{z_i \mid i \in U\}$, if every T_i is locally strongly monotone at z_i, with the same modulus α and with the same constant $\delta > 0$, under Definition 4.7.1.

(c) $\{T_i \mid i \in U\}$ are (locally) consistently Lipschitz continuous, with constant M (and bound η) at $\{z_i \mid i \in U\}$, if every T_i is locally Lipschitz continuous at z_i, with the same Lipschitz constant M and the same η under Definition 4.7.2.

Theorem 5.5.1. *Let $H_k \equiv S^2$ in (5.2.2) and z^* be a common zero point of T and T^k, $k = 0, 1, \ldots$. If one of the three conditions in Part (b) of Theorem 5.4.2 is locally satisfied at Sz^*, then*

(i) Sz^* *is a fixed point of every extended FBS operator* $E_k = (I + \lambda_k \bar{T}_2^k)^{-1}(I - \lambda_k \bar{T}_1^k)$ *and* $\{E_k\}$ *is locally uniformly contractive at* Sz^**, with a constant* $\theta < 1$*, where* $\bar{T}_j^k = S^{-1} T_j^k S^{-1}$*, for* $j = 1, 2$.

(ii) *There exists a* $\kappa > 0$*, such that the sequence* $\{z^k\}$ *generated through the extended FBS method (5.2.2) will Q-linearly converge to* z^**, with ratio* θ *for any starting point* z^0*, with* $\|z^0 - z^*\| \leq \kappa$.

Proof. Proposition 2.10.1 and Lemma 5.2.2 imply that Sz^* is a fixed point of every E_k. Similar to the proof of the part (a) of Theorem 4.7.5, or Theorem 4.7.6, or Theorem 4.7.7, we could prove the locally uniform contractivity of E_k at Sz^*, for $k \geq 0$, with a constant $\theta < 1$, if one of the three conditions is satisfied locally at Sz^*.

For the proof of Part (ii), let $w^k = Sz^k$. Then, (5.2.2) implies that $w^{k+1} = E_k(w^k)$ for $k = 0, 1, \ldots$. Theorem 4.7.4 shows that $\{w^k\}$ Q-linearly converges to Sz^* with ratio θ for some $\kappa_1 > 0$ and any $\|w^0 - Sz^*\| \leq \kappa_1$. Thus, (ii) is true with $\kappa = \kappa_1 / \|S\|$. $\square$

Theorem 5.5.2. *Suppose* $z^* \in \mathrm{zer}(T) \bigcap_k \mathrm{zer}(T^k)$*. If* $\{\bar{H}_k\}$ *in (5.2.7) converges to an invertible bounded linear operator* H' *and one of the following conditions is satisfied:*

(a) $\{\lambda_k\}$ *satisfies (4.2.3), in which* $\{\bar{H}_k^{-\frac{1}{2}} T_1^k \bar{H}_k^{-\frac{1}{2}}\}$ *is consistently Lipschitz continuous with constant* M*,* $\{\bar{H}_k^{-\frac{1}{2}} T_2^k \bar{H}_k^{-\frac{1}{2}}\}$ *is consistently strongly monotone with modulus* α *at* $\{\bar{H}_k^{\frac{1}{2}} z^*\}$*, respectively.*

(b) $\{\lambda_k\}$ *satisfies (4.2.11), in which* $\{\bar{H}_k^{-\frac{1}{2}} T_1^k \bar{H}_k^{-\frac{1}{2}}\}$ *is consistently Lipschitz continuous with constant* M*,* $\{\bar{H}_k^{-\frac{1}{2}} T_1^k \bar{H}_k^{-\frac{1}{2}}\}$ *and* $\{\bar{H}_k^{-\frac{1}{2}} T_2^k \bar{H}_k^{-\frac{1}{2}}\}$ *are consistently strongly monotone with moduli* α_1 *and* α_2*, respectively, at* $\{\bar{H}_k^{\frac{1}{2}} z^*\}$.

(c) $\{\bar{H}_k^{-\frac{1}{2}} T_1^k \bar{H}_k^{-\frac{1}{2}}\}$ *is consistently Lipschitz continuous with constant* M *and consistently strongly monotone with modulus* α *at* $\{\bar{H}_k^{\frac{1}{2}} z^*\}$*, and* $\{\lambda_k\}$ *satisfies (4.3.1).*

then

(i) $\bar{H}_k^{\frac{1}{2}} z^*$ *is a fixed point of extended forward-backward operator* $E_k = (I +$

$\lambda_k \bar{T}_2^k)^{-1}(I - \lambda_k \bar{T}_1^k)$ *and* $\{E_k\}_{k=0}^{\infty}$ *is locally consistently contractive with a constant* $\theta < 1$ *at* $\{\bar{H}_k^{\frac{1}{2}} z^*\}$, *where* $\bar{T}_j^k = \bar{H}_k^{-\frac{1}{2}} T_j^k \bar{H}_k^{-\frac{1}{2}}$ *for* $j = 1, 2$.

(ii) There exists a $\kappa > 0$, *such that the sequence* $\{z^k\}$, *generated through the extended FBS method (5.2.7), will strongly converge to* z^* *for any starting point* z^0 *with* $\|z^0 - z^*\| \leq \kappa$.

Proof. Proposition 2.10.1 and Lemma 5.3.3 imply that $\bar{H}_k^{\frac{1}{2}} z^*$ is a fixed point of E_k for $k = 0, 1, \cdots$. Similar to the proof of the part (a) of Theorem 4.7.5, or Theorem 4.7.6, or Theorem 4.7.7, we can prove the local consistent contractivity of $\{E_k\}$ at $\left\{\bar{H}_k^{\frac{1}{2}} z^*\right\}$ for $k \geq 0$, with a constant $\theta < 1$, if one of the three conditions in this theorem is satisfied.

Part (ii) may be proved thusly:

Let $R_k = \sqrt{\bar{H}_k}$ and $w^k = R_k z^k$. Then

$$w^{k+1} = R_{k+1} R_k^{-1} E_k(w^k).$$

Hence

$$w^{k+1} - R_{k+1} z^* = R_{k+1} R_k^{-1} (E_k(w_k) - E_k(R_k z^*)).$$

Since $\{E_k\}$ is locally consistently contractive at $\{R_k z^*\}$, there is a $\kappa_0 > 0$, such that

$$\|E_k(w) - E_k(R_k z^*)\| \leq \theta \|w - R_k z^*\|, \quad \forall\, k \geq 0 \;\; \& \;\; \|w - R_k z^*\| \leq \kappa_0. \tag{5.5.1}$$

Since $R_{k+1} R_k^{-1} \to I$ as $k \to \infty$, there is a $\bar{k} \geq 0$ for any given η with $\theta < \eta < 1$ such that

$$\theta \|R_{k+1} R_k^{-1}\| \leq \eta \qquad \text{when} \quad k \geq \bar{k}. \tag{5.5.2}$$

Let

$$\bar{C} = \prod_{k=0}^{\bar{k}-1} \|R_{k+1} R_k^{-1}\| \qquad \text{and} \qquad \kappa_1 = \kappa_0 / \bar{C}.$$

Then, by induction, (5.5.1) and (5.5.2) imply that $\|w^k - R_k z^*\| \leq \kappa_1$ whenever $\|w^0 - R_0 z^*\| \leq \kappa_1$. So, when $k > \bar{k}$ and $\|w^0 - R_0 z^*\| \leq \kappa_1$,

$$\begin{aligned} \|w^k - R_k z^*\| &\leq \eta \|w^{k-1} - R_{k-1} z^*\| \\ &\leq \eta^{k-\bar{k}} \|w^{\bar{k}} - R_{\bar{k}} z^*\|. \end{aligned}$$

This implies $w^k \to R'z^*$ as $k \to \infty$. Here $R' = \sqrt{H'}$.

Now, let $\kappa = \kappa_1/\|R_0\|$. Then $\|w^0 - R_0 z^*\| \le \kappa_1$ when $\|z^0 - z^*\| \le \kappa$. Therefore, whenever $\|z^0 - z^*\| \le \kappa$, $z^k = R_k^{-1} w^k$ converges $(R')^{-1}R'z^* = z^*$ strongly. $\square$

Theorem 5.5.3. *Suppose $\{z^k\}$ is generated from (5.2.2) for a given z^0, and $H_k = S^2$ with S symmetric positive definite. If $\bar{z}_k \in \mathrm{zer}(T^k)$ for $k = 0, 1, \ldots$, (5.4.15) is true, and one of the following three conditions is satisfied,*

(a) *Assume that $\{S^{-1}T_1^k S^{-1}\}$ is consistently Lipschitz continuous, with constant M, $\{S^{-1}T_2^k S^{-1}\}$ is consistently strongly monotone with modulus α, respectively, at $\{S\bar{z}_k\}$, and $\{\lambda_k\}$ satisfies (4.2.3).*

(b) *Assume that $\{S^{-1}T_1^k S^{-1}\}$ is consistently Lipschitz continuous with constant M, $\{S^{-1}T_1^k S^{-1}\}$ and $\{S^{-1}T_2^k S^{-1}\}$ are consistently strongly monotone, with moduli α_1 and α_2, respectively, at $\{S\bar{z}_k\}$, and $\{\lambda_k\}$ satisfies (4.2.11).*

(c) *Assume that $\{S^{-1}T_1^k S^{-1}\}$ is consistently Lipschitz continuous, with constant M, and consistently strongly monotone, with modulus α, at $\{S\bar{z}_k\}$, and $\{\lambda_k\}$ satisfies (4.3.1).*

then

(i) *$S\bar{z}_k$ is a fixed point of extended forward-backward operator $E_k = (I + \lambda_k \bar{T}_2^k)^{-1}(I - \lambda_k \bar{T}_1^k)$ and, $\{E_k\}_{k=0}^{\infty}$ is locally consistently contractive at $\{S\bar{z}_k\}$, with a constant $\theta < 1$ and a bound $\kappa_0 > 0$. Where, $\bar{T}_j^k = S^{-1}T_j^k S^{-1}$ for $j = 1, 2$.*

(ii) *if, in addition, there exists a scalar sequence $\{\tau_k\}$, such that for $k = 0, 1, \ldots$*

$$0 < \tau_k \le \tau_{k+1} \le 1 \quad \text{and} \quad \|S\bar{z}_k - S\bar{z}_{k+1}\| \le (\tau_{k+1} - \tau_k\theta)\kappa_0, \tag{5.5.3}$$

then, $\|Sz^k - S\bar{z}_k\| \le \tau_k\kappa_0$ and the sequence $\{z^k\}$ will strongly converge to z^*, for any starting point z^0, provided $\|Sz^0 - Sz^*\| \le \tau_0\kappa_0$.

Proof. Proposition 2.10.1 and Lemma 5.3.3 imply that $S\bar{z}_k$ is a fixed point of E_k for $k = 0, 1, \cdots$. Similar to the proof of the part (a) of Theorem 4.7.5, or Theorem 4.7.6, or Theorem 4.7.7, we can prove the local consistent contractivity of $\{E_k\}$ at $\{S\bar{z}_k\}$, with a constant $\theta < 1$ and a bound $\kappa_0 > 0$, if one of the three conditions in this theorem is satisfied.

Part (ii) may be proved as follows:

Let $w^k = Sz^k$ and $\bar{w}_k = S\bar{z}_k$. Then

$$w^{k+1} = E_k(w^k) \qquad \text{and} \qquad \bar{w}_k = E_k(\bar{w}_k).$$

Since $\{E_k\}$ is locally consistently contractive at $\{\bar{w}_k\}$, with constant θ and bound κ_0 from the part (i), therefore,

$$\|E_k(w) - E_k(\bar{w}_k)\| \le \theta\|w - \bar{w}_k\|, \quad \forall\ k \ge 0 \ \text{ and } \ \|w - \bar{w}_k\| \le \kappa_0. \tag{5.5.4}$$

By induction, if $\|w^k - \bar{w}_k\| \le \tau_k\kappa_0$, from (5.5.3) it follows that

$$\begin{aligned} \|w^{k+1} - \bar{w}_{k+1}\| &\le \|E_k(w^k) - E_k(\bar{w}_k)\| + \|\bar{w}_k - \bar{w}_{k+1}\| \\ &\le \theta\tau_k\kappa_0 + (\tau_{k+1} - \theta\tau_k)\kappa_0 = \tau_{k+1}\kappa_0. \end{aligned} \tag{5.5.5}$$

Therefore, the proof of the first conclusion in the part (ii) is complete.

Now, when $\|w^0 - \bar{w}_0\| \le \tau_0\kappa_0$, (5.5.3) and (5.5.5) imply that

$$\begin{aligned} \|w^{k+1} - \bar{w}_k\| &= \|E_k(w^k) - E_k(\bar{w}_{k-1}) + E_k(\bar{w}_{k-1}) - E_k(\bar{w}_k)\| \\ &\le \theta\|w^k - \bar{w}_{k-1}\| + \theta\|\bar{w}_{k-1} - \bar{w}_k\|. \end{aligned} \tag{5.5.6}$$

(5.4.15), (5.5.6), and Lemma 5.3.6 imply that

$$\lim_{k\to\infty} w^k = \lim_{k\to\infty} \bar{w}_k = Sz^*.$$

Hence $\{z^k\} = \{S^{-1}w^k\}$ converges strongly to z^*. $\square$

Now we shall consider a local convergence theorem under the conditions of Theorem 5.4.5. First the following definition is offered.

Definition 5.5.3. *Let $\{T_i \mid i \in U\}$ and $\{C_i \mid i \in U\}$ be, respectively, a set of operators and a set of closed sets with the index set U. We describe $\{T_i\}$ as (locally) consistently strongly monotone on $\{C_i\}$, with modulus α, if every T_i is strongly monotone on the closed set C_i, with the same modulus $\alpha > 0$.*

Let

$$\mathbb{B}(z, \delta) := \{w \mid \|w - z\| \le \delta\}.$$

Theorem 5.5.4. *Let $\{z^k\}$ be generated from (5.2.2) for a given z^0, and $H_k = S^2$ with S symmetric positive definite. Suppose that $\bar{z}_k \in \mathrm{zer}(T^k)$ for $k = 0, 1, \cdots$ and $\{S(T_1^k)^{-1}S\}$ is consistently strongly monotone on $\{\mathbb{B}(S\bar{z}_k, \kappa)\}$ with modulus β. If (2.10.9) is satisfied and $T^k \to T$ when $k \to \infty$ and*

$$\mu = \sum_{k=0}^{\infty} \|\bar{z}_{k+1} - \bar{z}_k\| < \kappa, \tag{5.5.7}$$

then,

(i) $S\bar{z}_k = E_k(S\bar{z}_k)$ with E_k in Theorem 5.5.3 and for any $k \geq 0$

$$\|y_1 - y_2\| \leq \|w_1 - w_2\| \quad \text{for} \quad y_j \in E_k(w_j) \quad \text{and} \quad w_j \in \mathbb{B}(S\bar{z}_k, \kappa); \tag{5.5.8}$$

(ii) $\mathrm{zer}(T) \neq \emptyset$;

(iii) $\{z^k\}$ will weakly converges to a zero of T for any $\|Sz^0 - S\bar{z}_0\| \leq \kappa - \mu$ and (5.4.18) is true.

Proof. Proposition 2.10.1 and Lemma 5.3.3 imply that $S\bar{z}_k$ is a fixed point of E_k for $k = 0, 1, \cdots$. Similar to the proof of Theorem 4.7.8, we may prove (5.5.8) when the conditions in this theorem is satisfied.

From Lemma 5.3.2, where $\{w^k\}$ is from (5.4.8), we only need to prove that $\{w^k\}$ has the same convergence properties as $\{z^k\}$ under the conditions in the theorem.

Let $\bar{T}_1^k = S^{-1}T_1^kS^{-1}$, $\bar{T}_2^k = S^{-1}T_2^kS^{-1}$, $x^k \in \bar{T}_1^k(w^k)$ and $y^{k+1} \in \bar{T}_2^k(w^{k+1})$ be such that

$$w^{k+1} + \lambda_k y^{k+1} = w^k - \lambda_k x^k. \tag{5.5.9}$$

Note that T_1^k or $\bar{T}_1^k$ are single-valued on $\mathbb{B}(\bar{z}_k, \kappa)$ or $\mathbb{B}(S\bar{z}_k, \kappa)$ under the conditions given in the theorem. Let $w_k^* = S\bar{z}_k$ and $y_k^* = -\bar{T}_1^k(w_k^*)$. Then, $y_k^* \in \bar{T}_2^k(w_k^*)$. Hence,

$$w^{k+1} - w_k^* + \lambda_k(y^{k+1} - y_k^*) = w^k - w_k^* - \lambda_k(x^k - \bar{T}_1^k(w_k^*)).$$

So, if $\eta = \min\{\inf\{\lambda_k\}, 2\beta - \sup\{\lambda_k\}\}$ and $\|w^k - w_k^*\| \leq \kappa$, then, $\eta > 0$ and

$$\|w^{k+1} - w_k^*\|^2 + \eta^2\|y^{k+1} - y_k^*\|^2 \leq \|w^k - w_k^*\|^2 - \eta^2\|x^k - \bar{T}_1^k(w_k^*)\|^2. \tag{5.5.10}$$

By induction, if $\tau_0 = \kappa - \mu$ and $\tau_{k+1} = \tau_k + \|w_k^* - w_{k+1}^*\|$, then, when $\|w^0 - w_0^*\| \leq \tau_0$,

$$\begin{aligned}\|w^{k+1} - w_{k+1}^*\| &\leq \|w^{k+1} - w_k^*\| + \|w_k^* - w_{k+1}^*\| \\ &\leq \|w^k - w_k^*\| + \|w_{k+1}^* - w_k^*\| \\ &\leq \tau_k + \|w_{k+1}^* - w_k^*\| = \tau_{k+1} \leq \kappa.\end{aligned}$$

Thus, when $\|w^0 - w_0^*\| \leq \tau_0$, (5.5.10) implies that

$$\begin{aligned}\|w^{k+1} - w_k^*\| &\leq \|w^k - w_k^*\| \\ &\leq \|w^k - w_{k-1}^*\| + \|w_{k-1}^* - w_k^*\| \\ &\leq \|w^1 - w_0^*\| + \sum_{i=1}^{k} \|w_{i-1}^* - w_i^*\| \\ &\leq \|w^1 - w_0^*\| + \mu.\end{aligned}$$

Since $\{\bar{z}_k\}$ is bounded from (5.5.7), S is a bounded operator and (5.5.7) is true, then, the above inequalities imply that $\{w^k\}$ and $\{w^{k+1} - w_k^*\}$ are bounded sequences. Hence, there exists a weak cluster point w^∞ and a subsequence $\{w^{k(j)}\}$ of $\{w^k\}$, such that

$$w^{k(j)} \to w^\infty \quad \text{weakly as} \quad j \to \infty. \tag{5.5.11}$$

Now adding (5.5.10) over $k = 1, 2, \ldots, m$ gives

$$\begin{aligned}&\|w^{m+1} - w_m^*\|^2 + \sum_{k=1}^{m} \left[\eta^2 \|y^{k+1} - y_k^*\|^2 + 2\langle w_{k-1}^* - w^k, w_{k-1}^* - w_k^*\rangle\right] \\ &\leq \|w^1 - w_0^*\|^2 + \sum_{k=1}^{m} \|w_{k-1}^* - w_k^*\|^2 - \eta^2 \sum_{k=1}^{m} \|x^k - \bar{T}_1^k(w^*)\|^2.\end{aligned}$$

The boundedness of $\{w^{k+1} - w_k^*\}$, (5.5.7), and the above imply that the two positive term series are convergent as $m \to \infty$. That is,

$$\sum_{k=1}^{\infty} \|y^{k+1} - y_k^*\|^2 < \infty, \qquad \sum_{k=1}^{\infty} \|x^k - \bar{T}_1^k(w_k^*)\|^2 < \infty.$$

Thus,

$$\lim_{k\to\infty} x^k - \bar{T}_1^k(w_k^*) = 0, \qquad \lim_{k\to\infty} y^{k+1} - y_k^* = 0. \tag{5.5.12}$$

Now (2.10.9), (5.5.9), (5.5.12), and Proposition 2.1.5 imply, for $u^k = \bar{T}_1^k(w^{k+1})$, that

$$\begin{aligned} \|u^k - x^k\| &\leq \beta^{-1}\|w^{k+1} - w^k\| \\ &= \beta^{-1}\lambda_k\|y^{k+1} + x^k\| \to 0 \quad \text{as} \quad k \to \infty. \end{aligned} \tag{5.5.13}$$

$T^k \to T$ implies that $\bar{T}^k = \bar{T}_1^k + \bar{T}_2^k \to \bar{T} = S^{-1}TS^{-1}$ pointwisely. Hence, for any $w \in \mathcal{H}$ and $x \in \bar{T}(w)$, there is a sequence $\{v^k\}$ with $v^k \in \bar{T}^k(w)$ such that $v^k \to x$. Thus, (5.5.11), (5.5.12), (5.5.13), and Lemma 4.1.8 give the following:

$$0 \leq \lim_{j\to\infty} \langle w^{k(j)} - w, y^{k(j)} + u^{k(j)-1} - v^{k(j)-1}\rangle = \langle w^{\infty} - w, -x\rangle \tag{5.5.14}$$

for any $x \in \bar{T}(w)$ and $w \in \mathcal{H}$. From the maximal monotonicity of T, and hence $\bar{T}$, (5.5.14) shows us that $0 \in \bar{T}(w^{\infty})$. Therefore, $z^{\infty} = S^{-1}w^{\infty}$ is a zero point of T. However, (2.10.9), (5.5.12) and

$$w^{k+1} = w^k - \lambda_k[y^{k+1} + x^k]$$

imply that $\{w^k\}$ actually converges weakly to w^{∞}. That is, $\{z^k\}$ converges weakly to z^{∞} with $0 \in T(z^{\infty})$. This completes the proof of Part (ii) and the first part of Part (iii).

Now, (5.5.9) and (5.5.12) imply that $\|w^{k+1} - w^k\| \to 0$, and hence $\|z^{k+1} - z^k\| \to 0$ as $k \to \infty$. Now,

$$\begin{aligned} \sum_{k=0}^{\infty} \|w^{k+1} - w^k\|^2 &= \sum_{k=0}^{\infty} \lambda_k^2\|y^{k+1} + x^k\|^2 \\ &\leq \sum_{k=0}^{\infty} 2\lambda_k^2\left[\|y^{k+1} - y_k^*\|^2 + \|x^k - \bar{T}_1^k(w_k^*)\|^2\right] < \infty \end{aligned}$$

since $y_k^* + \bar{T}_1^k(w_k^*) = 0$ and $0 < \lambda_k < 2\beta$. Hence, $w^k = Sz^k$ implies (5.4.18). □

5.6. Quadratic and Superlinear Convergence of Extended FBS Methods

Now, let us turn our attention to the higher convergent rates of the extended forward backward splitting methods.

Theorem 5.6.1 (Finite termination). *If $T = T_1^k + T_2^k$ is strongly monotone with α, then, when we take $T_1^k(x) = \alpha x + a_k$, with $a_k \in \mathcal{H}$ and $\lambda_k = \alpha^{-1}$, the EFBS scheme (5.2.2), with $H_k \equiv I$, will terminate at one iteration, for a zero point of T.*

Proof. In this situation, (5.2.2) becomes

$$z^{k+1} = (I + \alpha^{-1}T_2^k)^{-1}(I - \alpha^{-1}T_1^k)(z^k).$$

So, there are $y^k \in T(z^{k+1})$, for any $k \geq 0$, such that

$$z^{k+1} + \alpha^{-1}[y^k - T_1^k(z^{k+1})] = z^k - \alpha^{-1}T_1^k(z^k).$$

That is,

$$z^{k+1} + \alpha^{-1}y^k - z^{k+1} - \alpha^{-1}a_k = z^k - \alpha^{-1}a_k - z^k.$$

This implies $y^k = 0$ for all $k = 0, 1, \cdots$. So, when $k = 0$, we have $0 \in T(z^1)$. □

Theorem 5.6.2 (Finite termination). *If $T = T_1^k + T_2^k$ is with $T_1^0(x) = L_0x + a_0$ and L_0 is a strongly monotone and bounded linear operator, then, the EFBS scheme (5.2.2), with $H_0 = L_0$, will terminate at one iteration for a zero point of T if $\lambda_0 = 1$.*

Proof. It is easy to see from (5.2.2) and $T = T_1^0 + T_2^0$ that

$$z^1 = (H_0 + T_2^0)^{-1}(-a_0), \qquad T_2^0(z) = (T - L_0)(z) - a_0.$$

Hence, $-a_0 \in H_0z^1 + T_2^0(z^1) = L_0z^1 + (T - L_0)(z^1) - a_0$. That is, $0 \in T(z^1)$. Therefore, z^1 is a zero point of T. □

Suppose $\{z^k\}$ is a sequence with limit point z^*. Recall that $\{z^k\}$ converges *superlinearly* to z^*, if

$$\frac{\|z^{k+1} - z^*\|}{\|z^k - z^*\|} \to 0 \quad \text{as} \quad k \to \infty. \tag{5.6.1}$$

$\{z^k\}$ is said to have a *quadratic convergence* rate, or that it converges to z^* at a *square rate*, if there is a constant C, such that

$$\|z^{k+1} - z^*\| \leq C\|z^k - z^*\|^2. \tag{5.6.2}$$

Theorem 5.6.3 (Superlinear rate). *Suppose $z^* \in \mathrm{zer}(T)\bigcap_k \mathrm{zer}(T_k)$ and $\bar{H}_k \to H'$. Then, the $\{z^k\}$ from (5.2.7) will superlinearly converge to z^*, for any given initial z^0 under one of the following conditions:*

(a) Let M_k and α_k be the Lipschitz constant of $\bar{H}_k^{-\frac{1}{2}} T_1^k \bar{H}_k^{-\frac{1}{2}}$ and strong monotone modulus of $\bar{H}_k^{-\frac{1}{2}} T_2^k \bar{H}_k^{-\frac{1}{2}}$. Assume that $\lambda_k = \alpha_k / M_k^2$ and $\bar{H}_k$, T_1^k can be chosen such that $M_k \to 0$ as $k \to \infty$ in addition to the assumptions of part (b) of Theorem 5.4.3.

(b) Let M_k and α'_k be the Lipschitz constant and strong monotone modulus of $\bar{H}_k^{-\frac{1}{2}} T_1^k \bar{H}_k^{-\frac{1}{2}}$ and α''_k be the strong monotone modulus of $\bar{H}_k^{-\frac{1}{2}} T_2^k \bar{H}_k^{-\frac{1}{2}}$. Assume that $\lambda_k = (\alpha'_k + \alpha''_k)/(M_k^2 + \alpha'_k \alpha''_k)$ and $\bar{H}_k$, T_1^k can be chosen such that $(M_k - \alpha'_k) \to 0$ as $k \to \infty$ in addition to the assumptions of part (c) of Theorem 5.4.3.

(c) Let M_k and α_k be the Lipschitz constant and strong monotone modulus of $\bar{H}_k^{-\frac{1}{2}} T_1^k \bar{H}_k^{-\frac{1}{2}}$. Assume that $\lambda_k = \alpha_k / M_k^2$ and $\bar{H}_k$, T_1^k can be chosen such that $\alpha_k M_k^{-1} \to 1$ as $k \to \infty$ in addition to the assumptions of part (a) of Theorem 5.4.3.

Proof. Let $R_k = \sqrt{\bar{H}_k}$, $w^k = R_k z^k$, $R' = \sqrt{H'}$, and E_k be the same as that in Theorem 5.5.2.

Part (a) : Similar to the proofs of inequality (4.3.4), we have

$$
\begin{aligned}
\|z^{k+1} - z^*\| &\le \|R_k^{-1}\| \|E_k(w^k) - E_k(R_k z^*)\| \\
&\le \|R_k^{-1}\| \frac{\sqrt{1 + \lambda_k^2 M_k^2}}{1 + \alpha_k \lambda_k} \|w^k - R_k z^*\| \\
&= \|R_k^{-1}\| \frac{M_k}{\sqrt{\alpha_k^2 + M_k^2}} \|w^k - R_k z^*\| \\
&\le \|R_k^{-1}\| \|R_k\| \frac{M_k}{\sqrt{\alpha_k^2 + M_k^2}} \|z^k - z^*\|.
\end{aligned}
$$

$R_k \to R'$ and $M_k \to 0$, as $k \to \infty$, imply that

$$
\lim_{k \to \infty} \frac{\|z^{k+1} - z^*\|}{\|z^k - z^*\|} = 0.
$$

Part (b) : Similar to the proofs of inequality (4.3.5), we have

$$\begin{aligned}
\|z^{k+1} - z^*\| &\le \|R_k^{-1}\|\|E_k(w^k) - E_k(R_k z^*)\| \\
&\le \|R_k^{-1}\| \frac{\sqrt{1 + \lambda_k^2 M_k^2 - 2\alpha'_k \lambda_k}}{1 + \alpha''_k \lambda_k} \|w^k - R_k z^*\| \\
&= \|R_k^{-1}\| \frac{\sqrt{M_k^2 - (\alpha'_k)^2}}{\sqrt{(\alpha''_k)^2 + 2\alpha'_k \alpha''_k + M_k^2}} \|w^k - R_k z^*\| \\
&\le \|R_k^{-1}\|\|R_k\| \frac{\sqrt{M_k^2 - (\alpha'_k)^2}}{\sqrt{(\alpha''_k)^2 + 2\alpha'_k \alpha''_k + M_k^2}} \|z^k - z^*\|.
\end{aligned}$$

$R_k \to R'$ and $(M_k - \alpha'_k) \to 0$, as $k \to \infty$, imply that

$$\lim_{k\to\infty} \frac{\|z^{k+1} - z^*\|}{\|z^k - z^*\|} = 0.$$

$\square$

Part (c) : Similar to the proofs of inequality (4.3.3), we have

$$\begin{aligned}
\|z^{k+1} - z^*\| &\le \|R_k^{-1}\|\|E_k(w^k) - E_k(R_k z^*)\| \\
&\le \|R_k^{-1}\| \sqrt{\lambda_k^2 M_k^2 + 1 - 2\alpha_k \lambda_k} \|w^k - R_k z^*\| \\
&= \|R_k^{-1}\| \sqrt{1 - \left(\frac{\alpha_k}{M_k}\right)^2} \|w^k - R_k z^*\| \\
&\le \|R_k^{-1}\|\|R_k\| \sqrt{1 - \left(\frac{\alpha_k}{M_k}\right)^2} \|z^k - z^*\|.
\end{aligned}$$

$R_k \to R'$ and $\alpha_k / M_k \to 1$, as $k \to \infty$, imply that

$$\lim_{k\to\infty} \frac{\|z^{k+1} - z^*\|}{\|z^k - z^*\|} = 0.$$

Lemma 5.6.4. *Suppose a bounded linear operator S is symmetric and positive definite and T is a monotone operator. If $y_i = (S+T)^{-1}(x_i)$ for $i = 1, 2$, then*

$$\|y_1 - y_2\| \le \|S^{-1}\|\|x_1 - x_2\|. \tag{5.6.3}$$

Proof. $y_i = (S+T)^{-1}(x_i)$ iff $S^{\frac{1}{2}} y_i = (I + S^{-\frac{1}{2}} T S^{-\frac{1}{2}})^{-1}(S^{-\frac{1}{2}} x_i)$.

So

$$\|S^{\frac{1}{2}}(y_1 - y_2)\| \le \|S^{-\frac{1}{2}}(x_1 - x_2)\|.$$

But

$$\|y_1 - y_2\| \le \|S^{-\frac{1}{2}}\|\|S^{\frac{1}{2}}(y_1 - y_2)\|.$$

Therefore, (5.6.3) is true. $\square$

Theorem 5.6.5 (Quadratic rate). *Suppose $T = T_1^k + T_2^k$, with $\{T_1^k\}$ uniformly strongly monotone with α and uniformly Lipschitz continuous with M. For given $\lambda_k \equiv 1$, if T_1^k is twice differentiable and $H_k = \nabla T_1^k(z^k)$, then, the $\{z^k\}$ from (5.2.2) will converge to the zero point z^* of T at square rate, when z^0 is near z^*.*

Proof. In the Taylor expansion of T_1^k we obtain

$$T_1^k(z) = T_1^k(z') + \nabla T_1^k(z')(z - z') + O\big(\|z - z'\|^2\big) \tag{5.6.4}$$

If T_1^k is strongly monotone, with modulus α, then

$$\langle z - z', \nabla T_1^k(z')(z - z')\rangle + \|z - z'\|^2 O(z - z') \ge \alpha\|z - z'\|^2.$$

So, for any $\varepsilon > 0$ and nonnegative k, there is a $\delta_k > 0$, such that

$$O(z - z')| \le \varepsilon, \quad \text{when} \quad \|z - z'\| < \delta_k.$$

That is,

$$\langle z - z', \nabla T_1^k(z')(z - z')\rangle \ge (\alpha - \varepsilon)\|z - z'\|^2, \quad \text{when} \quad \|z - z'\| < \delta_k.$$

When $\|z - z'\| \ge \delta_k$, the above is still true since $\nabla T_1^k(z')$ is a linear operator and there is always a constant N, such that, $\|z - z'\| < N\delta_k$. So, $\nabla T_1^k(z')$ is positive definite for any $z' \in \mathcal{H}$. In other words, $\nabla T_1^k(z')$ is strongly monotone, with modulus $\alpha - \varepsilon$, for any $z' \in \mathcal{H}$ and $\varepsilon > 0$. Therefore, $\{\nabla T_1^k(z')\}$ is uniformly strongly monotone, with modulus α, for any $z' \in \mathcal{H}$.

Similarly, we may prove that $\{\nabla T_1^k(z')\}$ is uniformly Lipschitz continuous, with M, for any $z' \in \mathcal{H}$ from (5.6.4), or that $\{\nabla T_1^k(z')\}$ is a uniformly bounded linear operator set, with bound M, for any $z' \in \mathcal{H}$.

Again, from (5.6.4) and the Lipschitz continuity of T_1^k, we have

$$\|\nabla T_1^k(z')(z-z') + O\big(\|z-z'\|^2\big)\| \le M\|z-z'\|.$$

Hence, from the boundedness of $\nabla T_1^k(z')$,

$$O\big(\|z-z'\|^2\big) \le 2M\|z-z'\|.$$

So, for $\|z-z'\| \ge 1$, we have

$$O\big(\|z-z'\|^2\big) \le 2M\|z-z'\|^2. \tag{5.6.5}$$

When $0 \ne \|z-z'\| < 1$, there exists a N, such that $\|N(z-z')\| \ge 1$. So, placing $N(z-z')$ in the position of $z-z'$, in (5.6.5), still results in the same inequality. If $z' = z$, then (5.6.5) is obviously true.

Now, let z^* be the zero point of T, $z' = z^k$, and $z = z^*$ in the above. From (5.6.4), (5.6.5), and Lemma 5.6.4,

$$\begin{aligned}
\|z^{k+1} - z^*\| &\le \|H_k^{-1}\|\|H_k(z^k - z^*) - T_1^k(z^k) + T_1^k(z^*)\| \\
&= \|H_k^{-1}\|O\big(\|z^k - z^*\|^2\big) \\
&\le \|H_k^{-1}\|2M\|z^k - z^*\|^2 \\
&\le 2\alpha^{-1}M\|z^k - z^*\|^2.
\end{aligned} \tag{5.6.6}$$

The last inequality is reached since H_k is strongly monotone with α for any nonnegative k. When

$$\|z^0 - z^*\| < \tfrac{1}{2}\alpha M^{-1}, \qquad \text{or} \quad \mu = 2\alpha^{-1}M\|z^0 - z^*\| < 1 \tag{5.6.7}$$

it is easy to prove by induction from (5.6.6) and (5.6.7) that

$$\begin{aligned}
&2\alpha^{-1}M\|z^k - z^*\| \le \mu^{2^k} \qquad \text{and} \\
&\|z^k - z^*\| \le \mu^{2^k - 1}\|z^0 - z^*\|, \quad \text{for} \quad k \ge 1.
\end{aligned} \tag{5.6.8}$$

(5.6.8) implies that $\{z^k\}$ strongly converges to z^* when (5.6.7) is satisfied. Therefore, $\{z^k\}$ will converge at square rate from (5.6.6) when (5.6.7) is satisfied. $\square$

Theorem 5.6.6 (Superlinear rate). *Suppose* $T = T_1^k + T_2^k$ *in which* $\{T_1^k\}$ *is uniformly strongly monotone with* α *and uniformly Lipschitz continuous with* M. *If* T_1^k *is continuously differentiable and* z^* *is the zero point of* T, *then when* $\{\nabla T_1^k(z^*)\}$ *converges and we choose* H_k *with* $H_k - \nabla T_1^k(z^*) \to 0$, *then there must exist step sizes* $\{\lambda_k\}$ *with* $\lambda_k \to 1$, *such that the* $\{z^k\}$ *from (5.2.2) superlinearly converges to* z^* *for any initial* z^0.

Proof. In the proof of Theorem 5.6.5 through (5.6.6), if we change $O\big(\|z - z'\|^2\big)$ as $o(\|z - z'\|)$, then, similarly, we have

$$\|z^{k+1} - z^*\| \le \|H_k^{-1}\| \Big[\|(H_k - \lambda_k \nabla T_1^k(z^k))(z^k - z^*)\| + |o(\|z^k - z^*\|)|\Big] \quad (5.6.9)$$

If there exist step sizes λ_k, with $\lambda_k \to 1$, such that the $\{z^k\}$ from (5.2.2) strongly converges to z^*, for any initial z^0, then from $H_k - \nabla T_1^k(z^*) \to 0$ and $\lambda_k \nabla T_1^k(z^k) - \nabla T_1^k(z^*) \to 0$, we see

$$\|z^{k+1} - z^*\| \le |o(\|z^k - z^*\|)|.$$

This is required.

Now we need to prove there exist step sizes λ_k, with $\lambda_k \to 1$, such that the $\{z^k\}$, from (5.2.2), strongly converges to z^* for any initial z^0.

Since $\{\nabla T_1^k(z^*)\}$ is uniformly strongly monotone with modulus α, the limit operator is still strongly monotone with α. So, there are step sizes $\{\bar\lambda_k\}$ such that sequence $\{y^k\}$ from EFBS scheme (5.2.2), with $\{\bar\lambda_k\}$, strongly converges to z^* for any initial y^0 from Theorem 5.4.3. For any given $0 < \mu < 1$, $H_k - \nabla T_1^k(z^*) \to 0$ and $\nabla T_1^k(y^k) - \nabla T_1^k(z^*) \to 0$ imply that there is a positive integer J, such that

$$\|H_k^{-1}\| \Big[\|H_k - \nabla T_1^k(y^k)\| + \|y^k - z^*\|^{-1} \big|o\big(\|y^k - z^*\|\big)\big|\Big] \le \mu \quad \text{for } k \ge J. \quad (5.6.10)$$

Now, allow $z^0 = y^0$ and $\{z^k\}$ from (5.2.2) with the following λ_k,

$$\lambda_k = \begin{cases} \bar\lambda_k & \text{for } k \le J, \\ 1 & \text{otherwise.} \end{cases} \quad (5.6.11)$$

Then, we see $z^k = y^k$ for $k \le J + 1$, and from (5.6.9), (5.6.10) that

$$\|z^{k+1} - z^*\| \le \mu^{k+1-J} \|z^J - z^*\| \quad \text{for} \quad k > J.$$

This implies that $z^k \to z^*$ as $k \to \infty$. That is, $\{z^k\}$ here strongly converges to z^* with $\{\lambda_k\}$ from (5.6.11). However, the $\{\lambda_k\}$ from (5.6.11) obviously satisfies $\lambda_k \to 1$ as $k \to \infty$. This proof is required. $\square$

Corollary 5.6.7. *Suppose $T = T_1^k + T_2^k$, in which $\{T_1^k\}$ is uniformly strongly monotone with α and uniformly Lipschitz continuous with M. If T_1^k is continuously differentiable and z^* is the zero point of T, then when $\{\nabla T_1^k(z^*)\}$ converges and we can choose H_k and $\lambda > 0$ with $H_k - \lambda\nabla T_1^k(z^*) \to 0$, then there must exist step sizes $\{\lambda_k\}$ with $\lambda_k \to \lambda$ such that the $\{z^k\}$ from (5.2.2) superlinearly converges to z^* for any initial z^0.*

5.7. Relations with the Other Methods for the Fundamental Problem

In this section, we consider a couple relationships in the forward-backward splitting scheme as well as other methodologies. More relationships will be given in Section 9.4 when we discuss the problem of variational inequality.

Theorem 5.7.1 (Relations among splitting schemes). *If T_1 and T_2 are maximal monotone, then the forward-backward splitting operator $G = (I+T_2)^{-1}(I-T_1)$ is equivalent to the following double forward splitting operator, or double backward splitting operator, if $I - 2T_1$ is nonexpansive, or T_1^{-1} is strongly monotone with modulus 1,*

$$\begin{aligned} G &= (I+T_2)^{-1}(I-T_1) \\ &= (I+T_2)^{-1}(I+R_1)^{-1} \\ &= (I-R_2)(I-T_1) \end{aligned} \tag{5.7.1}$$

where $R_1 = (I-T_1)^{-1} - I = (T_1^{-1} - I)^{-1}$ and $R_2 = (I+T_2^{-1})^{-1}$ are maximal monotone.

Proof. If T_1 is maximal monotone and $I - 2T_1$ is nonexpansive, then there is a maximal monotone operator R_1, from Lemma 4.6.3, such that

$$T_1 = (I + R_1^{-1})^{-1} \quad \text{and} \quad I - T_1 = (I + R_1)^{-1}.$$

That is, $R_1 = (I-T_1)^{-1} - I = (T_1^{-1} - I)^{-1}$. Hence,

$$(I+T_2)^{-1}(I-T_1) = (I+T_2)^{-1}(I+R_1)^{-1}.$$

Let $R_2 = (I + T_2^{-1})^{-1}$. Then, R_2 is maximal monotone and $(I + T_2)^{-1} = I - R_2$ from Lemma 4.6.4. Therefore

$$(I + T_2)^{-1}(I - T_1) = (I - R_2)(I - T_1).$$

□

For a given x and $\lambda \neq 0$, let

$$\begin{aligned} \bar{x} &\in (I + \lambda T_2)^{-1}(I - \lambda T_1)(x) = G(x), \\ \bar{y} &\in (I + \lambda T_1)^{-1}[(I + \lambda T_2)^{-1}(I - \lambda T_1) + \lambda T_1](x) = D(x). \end{aligned} \tag{5.7.2}$$

That is, $\bar{x}$ comes from the FBS scheme and $\bar{y}$ is obtained by the Douglas-Rachford splitting scheme.

Theorem 5.7.2 (FBS and D-R schemes). *Suppose that T_1 is strongly monotone and single-valued. If $\bar{x} = \bar{y}$, then $x = \bar{x}$, and hence, $0 \in (T_1 + T_2)(x)$.*

Proof. If $\bar{x} = \bar{y}$, then from (5.7.2)

$$\bar{y} + \lambda T_1(\bar{y}) = \bar{x} + \lambda T_1(x).$$

This implies $T_1(\bar{y}) = T_1(x)$ from $\bar{x} = \bar{y}$. But, T_1 is strongly monotone, so there is a positive α, such that

$$\alpha \|\bar{y} - x\|^2 \leq \langle \bar{y} - x, T_1(\bar{y}) - T_1(x) \rangle = 0.$$

Hence, $\bar{y} = x$ or $\bar{x} = x$. Thus, (5.7.2) gives $0 \in (T_1 + T_2)(x)$. □