Theory Qual

Fall 2010

Please answer all 4 questions below.

1. You are given a 3-DNF formula φ and one of its terms t. The question is whether t is essential in φ, i.e., whether dropping t results in a formula that is no longer logically equivalent to φ.

   Show that this problem is NP-complete.

2. Suppose that you can decide whether a given Boolean circuit C has at least one input that is accepted, in time $2^{o(n)} \cdot m^{O(1)}$ where n denotes the number of variables of C and m the number of gates of C.

   Show that for every fixed integer $k \geq 1$ the same then holds for deciding whether a given Boolean circuit C on k blocks $x_1, x_2, \ldots, x_k$ of n Boolean variables each satisfies

   $$(\exists x_1)(\forall x_2)\ldots(Q x_k)\, C(x_1, x_2, \ldots, x_k),$$

   where $Q = \exists$ if k is odd and $Q = \forall$ otherwise.

3. You are given a network $G = (V, E)$ with source s and sink t. Each edge $e \in E$ is associated with an integer capacity c_e. The goal is to find a maximum s-t flow $f : E \to [0, \infty)$ of minimum cost $C(f)$, where $C(f) = \sum_{e \in E} (f(e))^2$.

   (a) Show that there may not be an integral solution, i.e., a solution where all flow values are integral.

   (b) Give an algorithm that finds an integral maximum s-t flow that minimizes the cost over all integral maximum s-t flows. Your algorithm should run in time polynomial in the size of G and the value of a maximum s-t flow.
   Hint: Modify the Ford-Fulkerson algorithm so that at every step it adds a unit amount of flow along a path of least marginal cost.

4. Suppose there exists a randomized polynomial-time Turing reduction from satisfiability to itself that makes a single query.

   Show that if the distribution of the query only depends on the length of the input, then satisfiability is in coNP/poly.

   Hint: Consider taking up the probability that the query is satisfiable in the advice.

G O O D L U C K ! !