Theory Qual

Fall 2012

Please answer all questions below.

1. We are given an urn with one black and one white ball. We run the following process until the urn has exactly n balls: we pick a ball uniformly at random from the urn and return it to the urn along with another ball of the same color. Derive the distribution of the number of white balls in the urn at the end of the process.

2. Let σ be a permutation of $[n] \doteq \{1, \ldots, n\}$ other than the identity permutation. Let $d(\sigma)$ denote the greatest common divisor of $\sigma(i) - i$ over $i \in [n]$. For example, if σ swaps two adjacent elements then $d(\sigma) = 1$.

   Given n, you are asked to compute the number of permutations σ of $[n]$ for which $d = k$, for every $k \in [n]$. Show how to do this in time polynomial in n.

3. Call a string e idempotent for a language L over an alphabet Σ if for all $x, y \in \Sigma^*$

   $$xey \in L \Leftrightarrow xeey \in L.$$

   Show that for every regular language L there exists a positive integer c such that for all $z \in \Sigma^*$, z^c is idempotent for L, where z^c denotes z concatenated c times.

4. Recall that $\mathcal{C}/a(n)$ denotes the class $\mathcal{C}$ with at most $a(n)$ bits of advice. Let us denote by P-uniform-SIZE$(s(n))$ the class of all languages for which there exists an algorithm A and a constant b such that, on input n, A runs in time n^b and produces a circuit of size at most $s(n)$ that decides L at length n. Note that the algorithm A and the constant b may depend on L.

   (a) Show that for every constant c, $\mathrm{P} \not\subseteq \mathrm{DTIME}(n^c)/(n-1)$.

   (b) Show that for every constant d, $\mathrm{P} \not\subseteq$ P-uniform-SIZE(n^d).

G O O D   L U C K ! !