Theory of Computation
Breadth Exam
Fall 1991

Directions: Answer any THREE of the five questions below.

1. Boxes numbered 1 through N are placed along a straight line. Initially, the lids on these boxes are closed. For $k = 2$ to N, the following procedure is repeated: Examine the boxes numbered $k, 2k, 3k, \ldots, \lfloor \frac{N}{k} \rfloor k$. If the lid of a box in this sequence is open, then close it; otherwise, if the lid is closed, then open it.

   For example, for $k = 2$, you would open the lids on all even numbered boxes. Then for $k = 3$, you would open the lid on box number 3, close box number 6, open box number 9, etc.

   After the process is complete, which of the boxes have their lids closed?

   Hint: Do an example with $N = 17$. At the end, you would find four boxes with closed lids.

2. A graph $G = (V, E)$ is *bipartite* if V can be partitioned into two parts X and Y such that all the edges connect a vertex of X to a vertex of Y, i.e., $E \subseteq X \times Y$. Give an $O(\mid E \mid)$ time algorithm for deciding if a given graph is bipartite. Your are free to choose any convenient representation for the input. State your choice clearly.

3. Prove that the following problem is NP-Complete: Given a graph $G = (V, E)$ with a positive edge-weight w_i for edge $e_i \in E$, and a positive integer W; is there a depth-first spanning tree of G with weight less than or equal to W? The weight of a tree is the sum of the weights of all its edges.

   Hint: Show that the problem is NP-Complete even when restricted to complete graphs.

4. What language is generated by the following context free grammar? (Here ϵ is the empty string). Give a careful explanation of your answer.

$$S \rightarrow SS \mid aSb \mid bSa \mid \epsilon$$

   A grammar is *ambiguous* if there are two distinct rightmost derivations of some string in the language. Is the above grammar ambiguous?

5. Let L be the language consisting of all well formed algebraic expressions constructed using the binary operators $+, \times$, the variables x, y, z and parentheses $(,)$. Thus $(x + y) \times z \in L$ but $(((\times \times \times))) \notin L$.

   (i) It is well known that L is not regular. Prove this.

   (ii) Suppose we replaced L by L', where each expression is given in postfix form. The first expression above would then be written as $x\ y + z\ \times$. Is L' regular? Prove your answer.