Foundations of Computing
Depth Examination
Fall 1998

Directions: Answer any four of the six questions below.

1. Let $L$ be the set of strings in $\{0,1\}^*$ with the same number of occurrences of 01 as 10. For example, the string 0101 has two occurrences of 01 and one occurrence of 10, so it is not in $L$. Is $L$ regular?

2. Recall the concept of a well-formed string of parentheses. Thus, for example, $(())()$ is well formed, but $((())$ is not. Consider the following algorithm for determining if a string $s$ composed of parentheses is well formed. Here we define EXCESS($s$) to be the number of ('s occurring in $s$, less the number of )'s.

   0. If $s$ is empty, it is well-formed.
   1. Find the smallest prefix $t$ of $s$, beginning with (, for which EXCESS($t$) $= 0$. (If none exists, the string is ill-formed.) If $t = s$, write $s = (u)$ and apply the algorithm recursively to $u$. If $t \neq s$, write $s = tu$ and apply the algorithm recursively to $t$ and $u$.

   Suppose the algorithm is applied to a string of length $n$. Give a big-O estimate for its running time.

3. Suppose that $X_1, ..., X_n$ is a random permutation of $1, ..., n$. Let $k$ be the length of the largest increasing sequence starting at $X_1$. For example, if $X_1 > X_2$, then $k = 1$.

   a) Show that the average value of $k$ is $O(1)$.
   b) Give an asymptotic value for this average.

4. Consider a Turing machine model with input alphabet $\{0,1\}$ in which inputs are infinite binary strings. Each input can be thought of as an infinite binary representation of a real number in the range $[0,1]$. Suppose also that the machine has an output tape which is write-only, that is, the head can never move to the left.

   We say that $f : [0,1] \to [0,1]$ is *computable* if there exists such a machine $M$ that, on any real input $x$, for all positive integers $i$, writes the $i$th bit of $f(x)$ (represented as an infinite binary string) in the $i$th cell of its output tape in a finite number of steps.

   Show that if $f : [0,1] \to [0,1]$ is computable in this sense, then $f$ is continuous.

   Notes: 1) A number can have more than one binary representation, for example $010000\ldots$ is the same as $001111\ldots$. It is sufficient that the machine produces *some* representation of $f(x)$. 2) The function $f$ is continuous at $x$ if, for any $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x) - f(x')| < \epsilon$ whenever $|x - x'| < \delta$.

5. Show that the problem of determining if an undirected bipartite graph has a perfect matching is log-space hard for the class NL (nondeterministic log space).

6. Each of $n$ elements is assigned one of two colors, red or blue. You want to determine if more than half of the elements are assigned one color and if so, identify one element of the majority color. You are allowed to repeatedly choose any two elements and ask whether they have the same color or not.

   a) Show that $n - 1$ questions are sufficient to solve this problem and that a linear number of questions are necessary to solve the problem.

   b) When $n = 7$ show that 4 questions are sufficient to solve the problem.

   c) Give an algorithm that solves the problem using a number of questions that is, in the worst case, at most

$$n - [\text{ the number of 1's in the binary representation of } n \text{ }].$$