Theoretical Computer Science Qualifying Exam

Fall 1999

Please do 4 problems from the 6 given below.

1. Let
$$DOUBLE-SAT \;=\; \{f : f \text{ has at least 2 satisfying assignments }\}.$$
Show that $DOUBLE-SAT$ is NP-complete.

2. Show that if P = NP we can factor integers in polynomial time. (Be careful. Remember that P and NP are language classes and factoring requires computing a function.)

3. Prove that if NEXPTIME $\neq$ EXPTIME, then P $\neq$ NP. Note: EXPTIME denotes $\bigcup_k DTIME(2^{n^k})$, and similarly for NEXPTIME.

4. Define the function $\mathrm{MAJORITY}_n : \{0,1\}^n \to \{0,1\}$ to be the function that returns the bit that occurs most in the input. (If exactly half the input bits are 1's you may assume the function returns a 1.)
   a. Show that $\mathrm{MAJORITY}_n$ can be computed with an $\mathrm{O}(n^2)$ size Boolean circuit.
   b. You may know that for each n there is an $\mathrm{O}(n)$ size circuit for adding two n bit numbers. Using this fact, show that $\mathrm{MAJORITY}_n$ can be computed with an $\mathrm{O}(n \log n)$ size circuit.

5. An undirected graph is k-colorable if each of its vertices can be assigned one of k colors, in such a way that no two adjacent vertices have the same color. Give a linear time algorithm to determine whether a graph is 2-colorable.

6. To estimate a function defined by a recurrence relation, it is often enough to consider only those integers which are powers of two. However, this is not always the case. The following recurrence relation is an exception. Let
$$F(n) \;=\; \begin{cases} F(n/2)+1, & \text{if } n \text{ is even;} \\ 2F(n/2)+1, & \text{if } n \text{ is odd.} \end{cases}$$
and $F(1) = 1$.
   a. Show that for powers of 2, $F(n)$ is $\mathrm{O}(\log n)$.
   b. Show that for infinitely many n, $F(n)$ is $\Omega(n)$.