Theory Qual

Spring 2004

Please answer 3 out of the 4 problems below.

1. Consider the problem where you are given an undirected graph G with integer weights on the edges, and an integer k. You would like to know whether G has a spanning tree of weight exactly k.

   Show that this problem is NP-complete.

2. Show that if we additionally restrict the weights in problem 1 to be in $\{0, 1\}$ then the problem is in P.

3. Recall that the permanent function perm over the integers is #P-hard. More specifically, there exists a polynomial-time computable transformation f that maps a given 3-CNF formula x into a square integer matrix $f(x)$ such that

   $$\mathrm{perm}(f(x)) = 12^m \cdot \#\mathrm{SAT}(x), \tag{1}$$

   where m represents the number of clauses in x and $\#\mathrm{SAT}(x)$ the number of satisfying assignments of x.

   (a) We know that even computing the permanent of 0-1 matrices is #P-hard. Nevertheless, argue that it is unlikely that we can restrict the range of the polynomial-time computable function f satisfying (1) to 0-1 matrices because the existence of such a stronger reduction would imply an unlikely collapse of complexity classes.

   (b) We call a function $g : \{0, 1\}^* \to \{0, 1\}^*$ *verifiable* if there exists a polynomial-time computable predicate $V : \{0, 1\}^* \times \{0, 1\}^* \to \{0, 1\}$ such that for any $x, y \in \{0, 1\}^*$, $V(x, y) = 1$ iff $g(x) = y$.

   Consider perm (using the standard binary encoding for integers). Show that perm is verifiable iff perm is computable in polynomial time.

   (c) Establish a similar result as (b) in case we only require V to be in BPP instead of in P.

4. Suppose that $G = (V, E)$ is a binary tree. The diameter of G is the maximum distance between any pair of vertices of G. (The distance between adjacent vertices is 1.)

   (a) Give a linear-time algorithm for computing the diameter of a binary tree.

   (b) Suppose that we construct n node binary search trees by randomly permuting n distinct keys and inserting them into a search tree. What is the average diameter for such trees? You can express your answer using asymptotic notation. Please prove your result.

G O O D   L U C K ! !