# Theory Qualifying Exam
## Spring 2010

**Directions.** You have four hours. There are 4 problems, please do them all. If you cannot completely solve a problem, we will award partial credit for work that is correct and relevant to the question.

**Problem 1.** Consider the following graph problem. Given an undirected graph with a collection of paths to a distinguished destination vertex, and an integer $k$. Are there $k$ paths that are edge disjoint?

Show that this problem is NP-complete.

**Problem 2.** Given a directed graph with weights on edges, a *monotone path* is a directed path in which every edge (except the first) has a weight strictly larger than that of the edge preceding it.

Give an algorithm for finding a monotone path with the largest number of edges in a given graph. Your algorithm should run in time polynomial in $n$ and $m$, the number of vertices and edges in the graph, respectively.

**Problem 3.** Suppose on a 2-way infinite array indexed by the integers $\mathbf{Z}$ there is a contiguous segment $S$ of length $m$. The segment occupies cells $\{c, c+1, c+2, \ldots, c+m-1\}$, where $m$ is known and $c$ is unknown.

We consider the following adversarial query model: We can query a cell $i$, and the answer is either YES or NO. If $i \notin S$ then the answer is NO. If $i \in S$, then the adversary can answer either YES (a true answer) or NO (which is a lie). The same query can be made multiple times and the adversary can lie at most $\ell$ times in total. To start, it is known that $0 \in S$.

Devise an algorithm that finds $c$ using at most $O(\ell + \log m)$ queries in the worst case.

**Problem 4.** Let $L$ be a language over $\{0,1\}$ and $M$ a polynomial-time deterministic oracle Turing machine such that for a fraction $1/n^{O(1)}$ of the strings $r$ of length $2^n$,

$$(\forall\, x \in \{0,1\}^n) M^r(x) = L(x).$$

1. Show that for the special cases where (i) $L =$ SAT and (ii) $L =$ TQBF the above implies that $L$ is in BPP.

2. Show that, in general, the above implies that $L$ can be decided by a randomized polynomial-time Turing machine $N$ with bounded error and advice of length $O(n \log n)$. *Hint:* First show that for any $n$ and $k$ there exists a set $S \subseteq \{0,1\}^n$ of size $O(k)$ such that either (i) the fraction of strings $r$ for which $M^r$ agrees with $L$ on $S$ is at most $1/2^k$, or (ii) for each $x \in \{0,1\}^n$, for at least $2/3$ of the strings $r$ for which $M^r$ agrees with $L$ on $S$, $M^r$ also agrees with $L$ on $x$.