Theory of Computation
Breadth Exam
Spring 1992

Directions: Answer any THREE of the five questions below.

1. Show that the following problem is NP-complete:

   Given a graph $G = (V, E)$, and an integer k, is there a spanning tree T of G such that T has at most k leaves?

   Hint: Try $k = 1$ or $k = 2$.

2. For each $k \geq 1$, exhibit a regular language L_k that can be defined using a finite automaton with k final states but cannot be defined using a finite automaton with fewer than k final states. Prove that your languages have the desired property.

3. Suppose that you are given a collection of n horizontal line segments positioned in the x-y plane. (Each line segment is given as a pair of endpoints.) Describe how to build a data structure to hold this information so that queries of the following type can be answered in $O(\log n)$ time. *Query:* Does the vertical segment $((x, y_1), (x, y_2))$ intersect one of the horizontal segments in the data structure?

   How would you retrieve all horizontal segments that intersect the given vertical segment. How long would it take?

4. Let A_n be a deterministic finite automaton with states $\{0, ..., n-1\}$ and alphabet $\{a, b\}$. The transitions of this automaton are

   $$\delta(i, a) = i + 1 \bmod n$$

   and

   $$\delta(i, b) = f(i)$$

   where f is an arbitrary function.

   Prove that if any such automaton has exactly one final state, then it is minimal.

5. For this problem you are to consider searching for records in a random way. To formalize this, consider an array A of integers indexed from 1 to $2n$, in which half of the values A_i are even and half are odd. Your goal is to find an i for which A_i is even. This is to be done by repeatedly generating a random number k uniformly from the range $1 \leq k \leq 2n$, and then examining A_k to check if it is even or odd. What is the expected number of random choices that must be made?

   Suppose you do many searches of this type. Over the long run, what percentage of array elements examined will be even and what percentage will be odd?