Theory of Computation
Depth Exam
Spring 1992

Directions: Answer any FOUR of the seven questions below.

1. Show that the following problems are NP-complete.

   Given a graph $G = (V, E)$, a subset L of V and an integer k, is there a spanning tree T of G such that

   (a) the set of leaves of T is L?

   (b) there are no leaves of T outside L?

   (c) T has at most k leaves?

2. The pancake problem is as follows. You are given a stack of n pancakes numbered from 1 to n, in an arbitrary order. You want to sort the pancakes so that the pancake number i is the i^{th} pancake from the bottom of the stack. The only operation you are allowed to do on the pancakes is a flip: insert a pancake flipper underneath some pancake in the stack and flip the stack of pancakes that lie above the flipper so that their order is reversed.

   Prove a linear time upper and lower bound on the number of flips needed to sort the pancakes.

3. Give the best possible estimate for the solution to the following recurrence:

$$T(n) = \sqrt{n}T(\lfloor\sqrt{n}\rfloor) + n.$$

   You may assume suitable initial conditions.

4. Can one construct a breadth first spanning tree of an undirected graph in NC? Justify a positive answer by giving an algorithm, and a negative answer by a suitable argument.

5. Give the best algorithm you can for selecting a random point in a 2-dimensional convex polytope. (The input data consists of n arbitrary points $x_1, \ldots, x_n$; the polytope is the interior of the convex hull of those points.) The random point is to be selected according to a uniform distribution; you can assume that a uniform random source is available for producing real numbers in the range (0,1).

   Hint: Do the case $n = 3$, a triangle, first.

6. Recall that a heap is a binary tree in which each node contains a key that is larger than any of its children. For this problem you can restrict attention to full (balanced) binary trees, having $n = 2^k - 1$ nodes.

   (a) Show that there are $H(n) = n!/\prod t_i$ different heaps on a set of n distinct keys, where t_i is the size of the heap rooted at the ith node.

(b) At least $n-1$ comparisons are required to build a heap, because the root is the maximum key. Improve this result to show that the number of comparisons required, for n large, is at least cn, where $c > 1$. (It is not difficult to show this for $c = 4/3$; the best known value of c is apparently 1.3644... .)

7. Consider a graph $G = (V, E)$. Let N be a matching in G, M a matching of maximum size, and $k = \mid M \mid - \mid N \mid$. Prove that N has a set of k vertex disjoint augmenting paths, at least one of length at most $n/k - 1$.

   Hint: Consider the symmetric difference of N and M.