Optimization

Fall 2007 Qualifying Exam
September 17, 2007
Instructions: Answer any 5 of the following 8 questions.

1. Consider the following linear program:

$$\begin{aligned} \max \quad & 4x_1 - x_2 \\ \text{subject to} \quad & 7x_1 - x_2 \leq 14 \\ & x_2 \leq 4 \\ & x_1 - x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

   (a) Find a primal optimal solution and a dual optimal solution. You may use any method you choose.

   (b) Is the primal optimal solution unique? Justify your response.

   (c) Is the dual optimal solution unique? Justify your response.

   (d) Assuming that no other data change and that the objective coefficient of x_2 changes from -1 to 0, will the primal solution that you identified remain optimal? If not, how do you know?

   (e) Under the same assumptions as in the previous questions, will the dual solution that you identified remain optimal? How do you know?

2. Consider the following linear program, with bounds and a single linear equality constraint:

$$\max \sum_{i=1}^{100} c_i x_i \text{ subject to } \sum_{i=1}^{100} a_i x_i = b, \quad 0 \le x_i \le u_i, \; i = 1, 2, \ldots, 100.$$

   (a) Write down the KKT optimality conditions for this problem.

   (b) Assume that $a_i > 0$ and $u_i > 0$ for all i, and that the variables are ordered such that

$$\frac{c_1}{a_1} \ge \frac{c_2}{a_2} \ge \ldots \ge \frac{c_{100}}{a_{100}}.$$

   Suppose further that

$$\sum_{i=1}^{50} a_i u_i + \frac{1}{2} a_{51} u_{51} = b.$$

   Using this information, find the primal solution x and the Lagrange multiplier vectors that satisfy the KKT conditions.

3. A manufacturing company has a permit to operate for T seasons, after which (in season $T+1$) it is only allowed to sell any leftover products. It is able to manufacture m different products, each requiring n different types of processing. Product i (for $i = 1, \ldots, m$) costs c_i dollars/liter to make and requires h_{ij} hours/liter of processing of type j (for $j = 1, \ldots, n$). Due to equipment limitations, the total time available for type j processing of all products during season t is H_{jt} hours (for $t = 1, \ldots, T$). (Potential complications about the order of processing are being ignored here.)

All the processing of an amount of product i must be completed in one season (it's not possible to start with some of the types of processing in one season and then finish with the others in the next), and that product can then be sold from the next season onward. To sell a liter of product i in season t requires e_{it} hours of marketing effort (labor). This labor can be hired in season t at the cost of d_t dollars/hour at the ordinary rate, up to a total of a_t hours. For additional labor beyond that, a higher rate of D_t dollars/hour must be paid. (There is no limit on the amount of hours at this higher rate or on the amounts of sales, which are regarded as a sure consequence of the marketing effort.) The selling price for product i in season t is p_{it} dollars/liter. If a quantity of product i is available for sale during a season, but is not sold then, the manufacturer has to pay q_i dollars/liter to store it and keep it in shape for possible sale in the next season. An alternative to marketing or storing is to donate quantities to charity. For this there is no monetary cost or reward. All products must be disposed of by the end of period $T+1$. Write a GAMS or AMPL model to determine what the manufacturer should do to maximize net profit over the entire period?

4. Let N be a directed network with two distinguished nodes s and t. Assume that there is a positive capacity on each arc and that there is at least one directed path from s to t as well as a return arc of unlimited capacity from t to s. Formulate the dual of the s-t max flow problem and show that it has an optimal solution in which all node prices have value 0 or 1. Discuss how this solution information can be used to obtain the min cut.

5. Let $G = (V, E)$ be an undirected graph with costs c_{ij} for each edge $(i, j) \in E$, and let $|V| = n$. Consider the following formulation of the symmetric traveling salesman problem:

$$
\begin{aligned}
\text{min} \quad & \sum_{(i,j)\in E} c_{ij}x_{ij} && (1)\\
\text{subject to} \quad & \sum_{j\in V:(i,j)\in E} x_{ij} = 2, \forall\, i \in V && (2)\\
& \sum_{(i,j)\in E: i\in S, j\in S} x_{ij} \le |S| - 1, \forall\, S \subset V,\ |S| \ge 3 && (3)\\
& x_{ij} \in \{0, 1\}, \forall (i, j) \in E && (4)
\end{aligned}
$$

Constraints (2) are the degree constraints; constraints (3) are the subtour elimination constraints.

(a) Consider the formulation in which we replace (2) by

$$\sum_{(i,j)\in E} x_{ij} = n. \qquad (5)$$

Does this this formulation have the same set of feasible solutions as the original? Either give a short proof that the set of feasible solutions is the same, or give a counterexample.

(b) Is it possible that the LP relaxation of the formulation (2), (5), (3), (4) might have extreme points that are fractional? How do you know?

(c) Now consider the LP relaxation of (2)–(4) and the LP relaxation of (2), (5), (3), (4). Do the two LP relaxations have the same set of feasible solutions?

6. (a) Prove that all isolated local minimizers of a function f are *strict* local minimizers.

(b) Suppose that we wish to form a quadratic model of a function $f : R^2 \to R$ by interpolating at six points. That is, we find the quadratic model q by enforcing the conditions $q(y^i) = f(y^i)$ for $i = 1, 2, 3, 4, 5, 6$. Show that if these six points are collinear, then q is not uniquely determined by the interpolation conditions.

7. Let f be a closed proper convex function on $\mathbb{R}^n$ and assume that $\inf f$ is finite.

   (a) Does this information imply that f attains a minimum on $\mathbb{R}^n$? If so, give a proof. If not, give a counterexample.

   (b) If you thought that the information above was insufficient to show that f attained a minimum, then explain the weakest additional assumption that you could make so that you could then prove attainment of the minimum, and give the proof using that assumption.

   You may use standard theorems of convex analysis, but if you do so then you must state the theorem you are using.

8. Suppose $F : \mathbf{R}^n \mapsto \mathbf{R}$ is a convex function and X is a nonempty closed convex set. Let

$$X^* = \{x^* \in X \mid F(x^*) \leq F(x), \forall x \in X\}.$$

   (a) Show that for every $y \in \mathbf{R}^n$ and $c > 0$ the minimum of $F(x) + (1/2c) \|x - y\|_2^2$ over $x \in X$ is attained at a unique point denoted by $x(y, c)$.

   (b) Show that the function $\Phi_c : \mathbf{R}^n \mapsto \mathbf{R}$ defined by

$$\Phi_c(y) = \min_{x \in X} \{F(x) + \frac{1}{2c} \|x - y\|_2^2$$

   is convex and that x^* minimizes $\Phi_c(y)$ over $y \in \mathbf{R}^n$ if and only if x^* minimizes $F(x)$ over $x \in X$, that is

$$X^* = \{x^* \mid \Phi_c(x^*) = \min_{y \in \mathbf{R}^n} \Phi_c(y)\}, \forall c > 0.$$

   Hint: you may assume that Φ_c is continuously differentiable and that its gradient is given by

$$\nabla \Phi_c(y) = \frac{y - x(y, c)}{c}.$$