# Fall 2011 Qualifier Exam:
# OPTIMIZATION

**September 19, 2011**

## GENERAL INSTRUCTIONS:

1. Answer each question in a separate book.
2. Indicate on the cover of *each* book the area of the exam, your code number, and the question answered in that book. On *one* of your books list the numbers of *all* the questions answered. *Do not write your name on any answer book.*
3. Return all answer books in the folder provided. Additional answer books are available if needed.

## SPECIFIC INSTRUCTIONS:

Answer 5 out of 8 questions.

## POLICY ON MISPRINTS AND AMBIGUITIES:

The Exam Committee tries to proofread the exam as carefully as possible. Nevertheless, the exam sometimes contains misprints and ambiguities. If you are convinced a problem has been stated incorrectly, mention this to the proctor. If necessary, the proctor can contact a representative of the area to resolve problems during the *first hour* of the exam. In any case, you should indicate your interpretation of the problem in your written answer. Your interpretation should be such that the problem is nontrivial.

1. Use the simplex method to solve the following problem:

$$\begin{array}{lcccccccc} \min & 4|x| & - & y & + & 3z & & \\ \text{s.t.} & & y & - & 2z & \leq & 4 \\ -x & - & y & + & 3z & \leq & -2 \\ 3x & - & y & + & 3z & \geq & 0 \\ & y & \geq & 3, & z & \leq & 1 \end{array}$$

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2. Let $a_1, \ldots, a_m \in \mathbb{R}^n$ be given vectors, and $b_1, \ldots, b_m \in \mathbb{R}$ be given scalars. Consider the problem:

$$z^* = \max \min_{i=1,\ldots,m} \{a_i x + b_i\} \tag{1a}$$

$$\text{s.t.} \sum_{j=1}^{n} x_j = 1 \tag{1b}$$

$$x_j \geq 0, \quad j = 1, \ldots, n. \tag{1c}$$

(a) Prove that there exists an optimal solution to (1) with at most $m$ nonzero values.

(b) Let $\bar{b} = \frac{1}{m}\sum_{i=1}^{m} b_i$ and $\bar{a}_j = \frac{1}{m}\sum_{i=1}^{m} a_{ij}$, for $j = 1, \ldots, n$. Prove:

$$z^* \leq \bar{b} + \max\{\bar{a}_j : j = 1, \ldots, n\}.$$

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3. Consider the problem

$$\min_{x_1, x_2 \geq 0} f(x) \stackrel{\text{def}}{=} -x_1 - x_2 + \mathbb{E}[F(x_1, x_2, \omega)],$$

where

$$F(x_1, x_2, \omega) = \min_{y \geq 0} \{\frac{1}{2} y \mid y \geq 1 - x_1 - \omega x_2\},$$

and $\omega$ is a random variable that has the following density:

$$\omega = \begin{cases} 1 & \text{with prob. } 0.25 \\ 2 & \text{with prob. } 0.5 \\ 4 & \text{with prob. } 0.25 \end{cases}$$

(a) Let $\hat{x} = (0, 0.5)^T$. Compute $\partial f(\hat{x})$.

(b) Compute a descent direction for $f$ at $\hat{x}$

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$$\min_{x_1, x_2 \geq 0} f(x) \stackrel{\text{def}}{=} -x_1 - x_2 + \mathbb{E}[F(x_1, x_2, \omega)],$$

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(a) Let $\hat{x} = (0, 0.5)^T$. Compute $\partial f(\hat{x})$.

(b) Compute a descent direction for $f$ at $\hat{x}$

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5. Consider a multicommodity flow problem on a graph with node set $V$ and arc set $A$, and let the set of commodities be $K$. For each $i \in V$ and $k \in K$, $b_{ik}$ is an integer representing the supply/demand of commodity $k$ at node $i$ (supply if positive, demand if negative). Given joint integer arc capacities $M_{ij}$ for $(i, j) \in A$, commodity-specific integral arc capacities $U_{ijk}$ for $(i, j) \in A$ and $k \in K$, and unit flow costs $c_{ijk} \geq 0$ for $(i, j) \in A$ and $k \in K$, the mulitcommodity minimum cost network flow problem is the following integer program:

$$z^* = \min \sum_{k \in K} \sum_{(i,j) \in A} c_{ijk} x_{ijk} \tag{2a}$$

$$\text{s.t.} \quad \sum_{j \in V^+(i)} x_{ijk} - \sum_{j \in V^-(i)} x_{jik} = b_{ik}, \qquad \forall i \in V, k \in K \tag{2b}$$

$$\sum_{k \in K} x_{ijk} \leq M_{ij}, \qquad \forall (i,j) \in A \tag{2c}$$

$$0 \leq x_{ijk} \leq U_{ijk}, \qquad \forall (i,j) \in A, k \in K \tag{2d}$$

$$x_{ijk} \in \mathbb{Z}, \qquad \forall (i,j) \in A, k \in K \tag{2e}$$

where $V^+(i) = \{j \in V : (i, j) \in A\}$ and $V^-(i) = \{j \in V : (j, i) \in A\}$. Let $z^{LP}$ be the optimal value of the LP relaxation of (2) (i.e., with (2e) replaced by $x_{ijk} \in \mathbb{R}$). Assume that (2) has a feasible solution.

(a) For any $u \in \mathbb{R}_+^{|A|}$, define the following Lagrangian relaxation of (2) in which the joint arc capacity constraints (2c) are relaxed:

$$w(u) = \min \sum_{k \in K} \sum_{(i,j) \in A} c_{ijk} x_{ijk} + \sum_{(i,j) \in A} u_{ij} \Big( \sum_{k \in K} x_{ijk} - M_{ij} \Big) \tag{3}$$

$$\text{s.t. } (2b), (2d), \text{ and } (2e)$$

Let $w^{LP}(u)$ be the optimal value of the LP relaxation of $w(u)$. Explain why $w^{LP}(u) = w(u)$.

(b) Now let $w^{LD} = \max\{w(u) : u \in \mathbb{R}_+^{|A|}\}$. How does $w^{LD}$ relate to $z^{LP}$? I.e., do we know (i) $w^{LD} = z^{LP}$, (ii) $w^{LD} \leq z^{LP}$ but equality may or may not hold, (iii) $w^{LD} \geq z^{LP}$ but equality may or may not hold, or (iv) we cannot in general compare $w^{LD}$ to $z^{LP}$? Justify your answer.

(c) Suppose for some $u \in \mathbb{R}_+^{|A|}$, we have solved (3) and obtained $\bar{x}$ as the optimal solution, $\bar{x}$ is feasible to (2c), and for any $(i,j) \in A$ with $u_{ij} > 0$, $\bar{x}$ satisfies the corresponding inequality in (2c) at equality, i.e.,:

$$\sum_{k \in K} \bar{x}_{ijk} = M_{ij}, \quad \forall (i,j) \in A \text{ such that } u_{ij} > 0.$$

Prove that $\bar{x}$ is an optimal solution to (2).

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6. Apply Newton's method with a constant stepsize to minimize $f(x) = \frac{1}{3}\|x\|_2^3$. Identify the range of stepsizes for which this method converges. Show that for any stepsize within this range, the iterates converge linearly to $x_{\text{opt}} = 0$. Determine the set of stepsizes in this range where the method converges superlinearly or explain why no such set exists.

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7. Let $C$ be a nonempty closed convex set in $\mathbb{R}^n$ and $A$ a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. Let $x_0 \in \mathbb{R}^m$, and suppose that $x_0$ does not belong to the set $A(C)$ (the image of $C$ under $A$).

Without further assumptions, can $\{x_0\}$ be *strongly* separated from $A(C)$ by a hyperplane? If your answer is yes, give a proof. If your answer is no, then

- Give a counterexample to the result without additional assumptions;
- State the weakest additional condition that you can find under which the result is true;
- Give a proof under that condition.

**Note:** Two nonempty convex sets $U$ and $V$ are strongly separated by a hyperplane $H$ if for some positive $\epsilon$, $H$ separates the sets $U + \epsilon B$ and $V + \epsilon B$, where $B$ is the unit ball.

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8. Consider

$$\begin{aligned} \theta = \quad & \min_{x \geq 0} \quad \textstyle\sum_{i=1}^n x_i[\log(x_i/u_i) + c_i] \\ & \text{s.t.} \quad x^T\mu \geq R \\ & \qquad\quad \textstyle\sum_i x_i + \text{card}(x)\beta + \sum_i \alpha_i x_i \leq B \end{aligned} \tag{4}$$

Here $u_i$, $\mu_i$, $R$, $\beta$, $\alpha_i$ and $B$ are positive data, but $c_i$ is arbitrary data.

(a) How would you solve this problem for $n \leq 10$?

(b) What makes this problem difficult for $n \geq 20$?

(c) Determine (and prove) a value for $\gamma$ so that

$$\begin{array}{ll} \min_{x \geq 0} & \sum_{i=1}^{n} x_i[\log(x_i/u_i) + c_i] \\ \text{s.t.} & x^T\mu \geq R \\ & \sum_i x_i + \beta\gamma \sum_i x_i + \sum_i \alpha_i x_i \leq B \end{array}$$

is a lower bound on $\theta$. Prove any properties of this problem that make it computationally tractable.

(d) How would you improve this bound if you know that $0 \leq x_i \leq \phi_i$?

(e) Describe a way to determine values for $\phi_i$ given a feasible point for (4).