Fall 2012 Qualifier Exam:
OPTIMIZATION

September 24, 2012

GENERAL INSTRUCTIONS:

1. Answer each question in a separate book.
2. Indicate on the cover of *each* book the area of the exam, your code number, and the question answered in that book. On *one* of your books list the numbers of *all* the questions answered. *Do not write your name on any answer book.*
3. Return all answer books in the folder provided. Additional answer books are available if needed.

SPECIFIC INSTRUCTIONS:

Answer 4 out of 6 questions.

POLICY ON MISPRINTS AND AMBIGUITIES:

The Exam Committee tries to proofread the exam as carefully as possible. Nevertheless, the exam sometimes contains misprints and ambiguities. If you are convinced a problem has been stated incorrectly, mention this to the proctor. If necessary, the proctor can contact a representative of the area to resolve problems during the *first hour* of the exam. In any case, you should indicate your interpretation of the problem in your written answer. Your interpretation should be such that the problem is nontrivial.

1. Consider the following convex quadratic program with a single equality constraint, non-negativity constraints, and a diagonal Hessian:

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} x^T Q x + c^T x \text{ subject to } a^T x = 1, \;\; x \geq 0, \tag{1}$$

where $a \in \mathbb{R}^n$ is a vector with all positive entries, and Q is a diagonal matrix with all positive diagonal entries.

(a) Suppose we drop the bounds $x \geq 0$ from the formulation (1). Write down the KKT conditions for the resulting simplified problem, and use them to deduce the solution x in closed form.

(b) Returning to the full problem (1), write down the KKT conditions, denoting the Lagrange multiplier for the constraint $a^T x = 1$ by λ.

(c) Fixing the value of λ in these KKT conditions, find the value of x_i, $i = 1, 2, \ldots, n$ that satisfies these conditions as an explicit function of λ. (Use the notation $x_i(\lambda)$, $i = 1, 2, \ldots, n$ to denote these values.)

(d) Show that the function $t : \mathbb{R} \to \mathbb{R}$ defined by

$$t(\lambda) = a^T x(\lambda) - 1 = \sum_{i=1}^{n} a_i x_i(\lambda) - 1$$

is a monotonic piecewise linear function of λ, and identify the breakpoints of this function (the points where the slope changes discontinuously).

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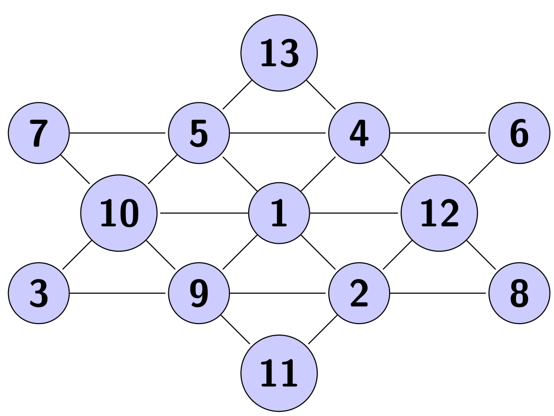
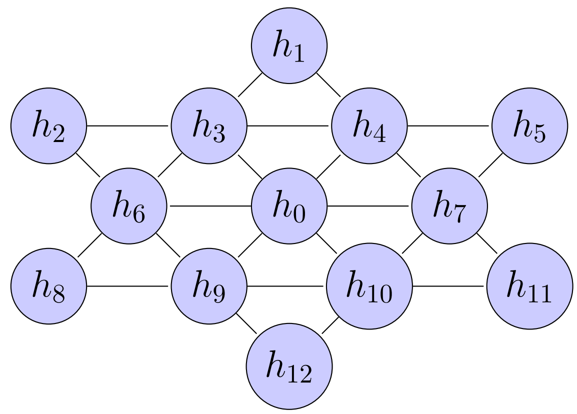
2. The Christmas board game "22" involves a board with 13 holes and 13 pegs which fit in the holes. The pegs are numbered from 1 to 13. Holes are situated at the 12 intersection points on a six-pointed star and in the center of the star. To play the game, a peg is inserted in each hole. A winning configuration is one in which the sum of values for each of the six outer triangles sums to 22. Here, for example, is a winning assignment:



In your solution, use the following indexing scheme to reference the game board holes:



(a) Determine which variables are needed to provide a solution to the game?

(b) Define a mapping $H(t)$ that provides the subset of "holes" used in triangle t and use this to write down the "22" constraint. Note that t will range from 1 to 6, indicating each of the "outer" triangles.

(c) Write the full mathematical (or GAMS) model which finds a solution to the game.

(d) Suppose this model is solved for one solution. Determine an additional constraint that would eliminate just this solution, and enable the model to be rerun to find another solution.

(e) What techniques could you use to remove "equivalent solutions" from within your model search? Provide two constraints that remove such "symmetries" from your search.

(f) Write pseudo-code (GAMS or similar for example) that shows the sequence of model solves that will find all solutions to the game.

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3. In this problem, we will consider the feasible region of a *chance-constrained* problem:

$$X = \{x \in \mathbb{R}^n \mid \mathbb{P}[g_i(x,\xi) \geq 0 \ \forall i = 1,\ldots,m] \geq 1-\epsilon\},$$

with each constraint function

$$g_i : \mathbb{R}^n \times \mathbb{R}^d \to \bar{\mathbb{R}},$$

and ξ being a random vector on a probability space $(\Omega, \Sigma, \mathbb{P})$.

(a) The set X is in general not convex. Give a simple example of constraints $g_i(x,\xi)$ and probability space $(\Omega, \Sigma, \mathbb{P})$, where X is *not* a convex set. Prove that your example set X is not convex.

(b) Now suppose the feasible region takes the form

$$X = \{x \in \mathbb{R}^n \mid \mathbb{P}(\xi^T x \geq b) \geq 1 - \epsilon\},$$

where $\xi \in \mathbb{R}^n$ is a **normally distributed** random vector with mean μ and covariance matrix Q. Show that X is convex if $\epsilon < 0.5$.

(c) In this concrete example, we will consider a production/distribution problem with a set J of customers whose (random) demand $d_j(\xi)$ must be met from a set of facilities I. Let x_{ij} be the amount of product shipped from $i \in I$ to $j \in J$. Suppose that the random demand for customer j comes from a discrete distribution; namely, that the demand of customer j in scenario $s \in S$ is d_{js} with probability p_s, for a finite set of scenarios S. We must choose the distribution amounts x_{ij} before the demands d_{js} are known. We would like to impose the constraint that the probability that *all* customers get their demand met is at least $1 - \epsilon$. Demonstrate how to model this using binary variables.

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4. Let X be the set of $(x, y) \in \{0,1\}^n \times \{0,1\}^{n(n-1)/2}$ that satisfy

$$y_{ij} \leq x_i, \quad \forall 1 \leq i < j \leq n \tag{2}$$

$$y_{ij} \leq x_j, \quad \forall 1 \leq i < j \leq n \tag{3}$$

$$x_i + x_j - y_{ij} \leq 1, \quad \forall 1 \leq i < j \leq n. \tag{4}$$

(a) Use Gomory-Chvátal rounding to show that the inequality

$$x_i + x_j + x_k \leq y_{ij} + y_{jk} + y_{ik} + 1 \tag{5}$$

is valid for conv(X) for any $1 \leq i < j < k \leq n$.

(b) Show that the inequality

$$y_{ij} + y_{ik} \leq x_i + y_{jk} \tag{6}$$

is valid for conv(X) for any $1 \leq i < j < k \leq n$. (You do not have to use Gomory-Chvátal rounding for this question, but you may if you wish.)

(c) Now, suppose that $n = 3$. Prove the specific case of inequality (6),

$$y_{12} + y_{13} \leq x_1 + y_{23}, \tag{7}$$

is facet-defining for conv(X). You may take as given the fact that dim(conv(X)) $= 6$, i.e., conv(X) is full-dimensional.

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5. (a) Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is convex and concave. Show that f must be an affine function.

(b) Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is convex and bounded above. Show that f must be a constant function.

(c) Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is strongly convex and Lipschitz, meaning that there is a constant L such that $|f(x) - f(y)| \le L||x - y||$ for all x and y. Show no such f exists.

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6. Consider the following optimization problem, which is parametrized by the scalar α:

$$P(\alpha): \qquad \min_{x \in \mathbb{R}^n} f(x) \text{ subject to } p^T x \le \alpha, \tag{8}$$

where $f : \mathbb{R}^n \to R$ is a smooth, strongly convex function and p is a nonzero vector in $\mathbb{R}^n$. We denote the optimal objective value for this problem by $\phi(\alpha)$, and note that the problem (8) has a unique minimizer $x(\alpha)$ for each $\alpha \in \mathbb{R}$.

(a) Show that ϕ is a convex, decreasing function of α.

(b) Show that ϕ is a continuous function of α.

(c) Show that there is a threshold value $\bar{\alpha}$ such that $\phi(\alpha) = \phi(\bar{\alpha})$ for all $\alpha \ge \bar{\alpha}$ while $\phi(\alpha) > \phi(\bar{\alpha})$ for all $\alpha < \bar{\alpha}$. (Hint: Consider the unconstrained global minimizer x^* of $f(x)$.)

(d) Consider the following related problem, in which $\lambda \ge 0$ is a parameter:

$$\min_{z \in \mathbb{R}^n} f(z) + \lambda p^T z, \tag{9}$$

where f and p are the same as in (8). Show that the point $z(\lambda)$ that solves (9) is identical to the solution $x(\alpha)$ of (8) if we set $\alpha = p^T z(\lambda)$.
