Fall 2013 Qualifier Exam: OPTIMIZATION

September 16, 2013

GENERAL INSTRUCTIONS:

1. Answer each question in a separate book.
2. Indicate on the cover of *each* book the area of the exam, your code number, and the question answered in that book. On *one* of your books list the numbers of *all* the questions answered. *Do not write your name on any answer book.*
3. Return all answer books in the folder provided. Additional answer books are available if needed.

SPECIFIC INSTRUCTIONS:

Answer all 4 questions.

POLICY ON MISPRINTS AND AMBIGUITIES:

The Exam Committee tries to proofread the exam as carefully as possible. Nevertheless, the exam sometimes contains misprints and ambiguities. If you are convinced a problem has been stated incorrectly, mention this to the proctor. If necessary, the proctor can contact a representative of the area to resolve problems during the *first hour* of the exam. In any case, you should indicate your interpretation of the problem in your written answer. Your interpretation should be such that the problem is nontrivial.

1. In this problem, we will attempt to solve the quadratic program:

$$\min_{x \geq 0} f(x) \stackrel{\text{def}}{=} \frac{1}{2} x^T M x + (r - ae)^T x \tag{P}$$

where we are given parameters $a \in \mathbb{R}_+, r \in \mathbb{R}^n$, $e \in \mathbb{R}^n$ is a vector of all ones, and the matrix M has the value 2 for its diagonal elements, and 1 everywhere else. Specifically, $M = J + I$, where $J = ee^T \in \mathbb{R}^{n \times n}$ is the matrix of all ones, and $I \in \mathbb{R}^{n \times n}$ is the identity matrix. We will assume throughout, without loss of generality, that $r_1 \leq r_2 \leq \ldots \leq r_n$.

   (a) Show that $M = J + I$ is positive-definite.

   (b) Write the KKT conditions for (P).

   (c) Establish the following monotonicity property for an optimal solution x^* to (P):

   $$x_i^* = 0 \Rightarrow x_j^* = 0 \; \forall j > i.$$

   Hint: You will need to use the fact that $r_1 \leq r_2 \leq \ldots \leq r_n$.

   (d) Show that

   $$M^{-1} = \frac{1}{n+1} \begin{bmatrix} n & -1 & \cdots & -1 \\ -1 & n & \ldots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n \end{bmatrix}$$

   (e) Let $\mathcal{R}$(P) be the problem (P) without the non-negativity constraints $x \geq 0$. Write a closed-form solution (formula) for an optimal solution to $\mathcal{R}$(P).

   (f) Assume that the first p components of the solution x^* to (P) are nonzero. (Note that the property (c) is satisfied by a solution with this structure.) Write down an explicit solution for these nonzero components. *Hints: The KKT conditions from (b) will yield a system of linear equations for the nonzero components, and part (d) will be useful in obtaining an explicit solution to this system.* **Note: Part (f) may be challenging. Please budget your time, ensuring that you have enough time for all parts of the exam.**

2. Consider the multi-item lot-sizing problem formulated below:

$$\min_{x,y,s} \sum_{i=1}^{m}\sum_{t=1}^{n}(p_{it}x_{it} + h_{it}s_{it} + f_{it}y_{it}) \tag{1a}$$

$$\text{s.t.} \sum_{i=1}^{m} C_{it}y_{it} \le B_t, \quad t = 1,\dots,n \tag{1b}$$

$$s_{i,t-1} + x_{it} - s_{it} = d_{it}, \quad t = 1,\dots,n,\ i = 1,\dots,m \tag{1c}$$

$$x_{it} \le D_{itn}y_{it}, \qquad t = 1,\dots,n,\ i = 1,\dots,m \tag{1d}$$

$$s_{it} \ge 0, x_{it} \ge 0, y_{it} \in \{0,1\},\ t = 1,\dots,n,\ i = 1,\dots,m. \tag{1e}$$

where $s_{i0} = 0$ is a constant, all data (B_i, C_{it}, d_{it}, p_{it}, h_{it}, f_{it}) is positive and $D_{it\ell} = \sum_{j=t}^{\ell} d_{ij}$ for any $1 \le t \le \ell$. For each item i, define

$$X_i = \{x_i \in \mathbf{R}_+^n, y_i \in \{0,1\}^n, s_i \in \mathbf{R}_+^n : (1c) - (1d)\}.$$

For each i, the ℓ-S inequalities

$$\sum_{j\in S} x_{ij} \le \sum_{j\in S} D_{ij\ell}y_{ij} + s_{i\ell}, \quad \forall S \subseteq \{1,\dots,\ell\}, \ell \in \{1,\dots,n\} \tag{2}$$

are valid for $\text{conv}(X_i)$. Furthermore, it is known that the convex hull of X_i is given by

$$\text{conv}(X_i) = \{x_i \in \mathbf{R}_+^n, y_i \in [0,1]^n, s_i \in \mathbf{R}_+^n : (1c) - (1d), (2)\}.$$

Let z_1^{LP} be the value of the LP relaxation of (1) (i.e., where $y_{it} \in \{0,1\}$ is replaced by $y_{it} \in [0,1]$) and let z_2^{LP} be the value of the LP relaxation of (1) augmented with the additional inequalities (2).

(a) Write down the Lagrangian relaxation problem obtained by relaxing constraints (1b) in the MIP formulation (1) and the associated Lagrangian dual problem.

(b) Can the Lagrangian relaxation problem from part (a) be solved by solving a set of smaller subproblems? If so, describe in words what these subproblems represent. If not, explain why not.

(c) Let w_1^{LD} be the optimal value of the Lagrangian dual associated with the relaxation you wrote in part (a). How does w_1^{LD} compare to z_1^{LP}? Choose one answer and explain. Note: here and in the following sub-questions, you may apply (without proof) known Lagrangian duality theory in your explanation.

   i. $w_1^{LD} = z_1^{LP}$.
   ii. $w_1^{LD} \ge z_1^{LP}$ and inequality might be strict.
   iii. $w_1^{LD} \le z_1^{LP}$ and inequality might be strict.

iv. Based on the given information either $w_1^{LD} > z_1^{LP}$ or $w_1^{LD} < z_1^{LP}$ is possible.

(d) How does w_1^{LD} compare to z_2^{LP}? Choose one answer and explain:

i. $w_1^{LD} = z_2^{LP}$.
ii. $w_1^{LD} \geq z_2^{LP}$ and inequality might be strict.
iii. $w_1^{LD} \leq z_2^{LP}$ and inequality might be strict.
iv. Based on the given information either $w_1^{LD} > z_2^{LP}$ or $w_1^{LD} < z_2^{LP}$ is possible.

(e) Now consider an alternative Lagrangian relaxation problem in which constraints (1c) are relaxed in the MIP formulation (1) (and (1b) are not relaxed). Let w_2^{LD} be the optimal value of the associated Lagrangian dual problem. How does w_2^{LD} compare to z_2^{LP}? Choose one answer and explain:

i. $w_2^{LD} = z_2^{LP}$.
ii. $w_2^{LD} \geq z_2^{LP}$ and inequality might be strict.
iii. $w_2^{LD} \leq z_2^{LP}$ and inequality might be strict.
iv. Based on the given information either $w_2^{LD} > z_2^{LP}$ or $w_2^{LD} < z_2^{LP}$ is possible.

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3. Let f be twice continuously differentiable. Suppose that x^* is a local minimum such that for all x in an open sphere S centered at x^*, we have, for some $m > 0$,

$$m \|d\|^2 \leq d^T \nabla^2 f(x) d, \quad \forall d \in \mathbb{R}^n.$$

Show that for every $x \in S$, we have

$$\|x - x^*\| \leq \frac{\|\nabla f(x)\|}{m}, \quad f(x) - f(x^*) \leq \frac{\|\nabla f(x)\|^2}{m}.$$

Hint: use the relation

$$\nabla f(y) = \nabla f(x) + \int_0^1 \nabla^2 f(x + t(y - x))(y - x) dt.$$

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4. Consider the following constrained optimization problem

$$\textbf{(A)} \quad \min_x f(x) \text{ s.t. } c_i(x) \geq 0, \quad i = 1, 2, \ldots, m,$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $c_i : \mathbb{R}^n \to \mathbb{R}$ are smooth functions. Consider the following reformulation of (A) that makes use of "squared slack variables:"

$$\textbf{(B)} \quad \min_{x,s} f(x) \text{ s.t. } c_i(x) - s_i^2 = 0, \quad i = 1, 2, \ldots, m.$$

(a) Write down the Karush-Kuhn-Tucker (KKT) conditions for both (A) and (B).

(b) If x^* is a KKT point for (A), verify that we can obtain a KKT point for (B) by setting x^* to the same value and defining $s_i^* = \sqrt{c_i(x^*)}$.

(c) If (x^*, s^*) is a KKT point for (B), is it true that x^* must be a KKT point for (A)? Explain.

(d) Write down the linear independence constraint qualification (LICQ) conditions for both (A) and (B). (Use $\mathcal{A}$ to denote the set of active indices in (A), that is, $\mathcal{A} := \{i = 1, 2, \ldots, m : c_i(x^*) = 0\}$.)

(e) Given a KKT point x^* for (A) at which LICQ holds, is it necessarily true that the LICQ conditions for (B) are satisfied at the corresponding KKT point for (B)?

(f) Write down the Mangasarian-Fromovitz constraint qualifications (MFCQ) for both (A) and (B).

(g) Given a KKT point x^* for (A) at which MFCQ holds, is it necessarily true that the MFCQ conditions for (B) are satisfied at the KKT point for (B) constucted as in part (b)?