Fall 2014 Qualifier Exam:
OPTIMIZATION

September 15, 2014

GENERAL INSTRUCTIONS:

1. Answer each question in a separate book.
2. Indicate on the cover of *each* book the area of the exam, your code number, and the question answered in that book. On *one* of your books list the numbers of *all* the questions answered. *Do not write your name on any answer book.*
3. Return all answer books in the folder provided. Additional answer books are available if needed.

SPECIFIC INSTRUCTIONS:

Answer 4 of 5 questions.

POLICY ON MISPRINTS AND AMBIGUITIES:

The Exam Committee tries to proofread the exam as carefully as possible. Nevertheless, the exam sometimes contains misprints and ambiguities. If you are convinced a problem has been stated incorrectly, mention this to the proctor. If necessary, the proctor can contact a representative of the area to resolve problems during the *first hour* of the exam. In any case, you should indicate your interpretation of the problem in your written answer. Your interpretation should be such that the problem is nontrivial.

1. Let $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$, $i = 1, \ldots, m$, $d > 0$, and $c \in \mathbb{R}^n$.

   (a) Formulate the problem

$$\max\Big\{c^\top x : \sum_{i=1}^{m} \max\{a_i^\top x - b_i, 0\} \le d, x \ge 0\Big\} \tag{1}$$

   as a compact linear program. (Compact here means that the number of decision variables and constraints is polynomial in n and m.)

   (b) Now write down an (exponential) number of linear inequalities just involving x that would be equivalent to the original nonlinear inequality in (1), and show this equivalence. (Hints: (i) To build intuition on what the form of this formulation will be, you may find it helpful to consider a special case with $n = 1$, and also start with $m = 1$ and $m = 2$. (ii) Alternatively, it may help to think about how you would check if a given solution $\hat{x}$ is feasible to (1).)

   (c) Explain how you would use a cutting plane approach for solving the formulation defined using the inequalities in part (b).

2. Consider the following data describing hydrological characteristics of a small hydroelectric power station.

   m denotes month

   f_m Water inflow in month m (million cubic meters)

   $\bar{p}_m$ Market price of electricity in month m

   L_{max} Maximum water level the dam can store (million cubic meters)

   L_{min} Minimum water level the dam can store (million cubic meters)

   R_{max} Maximum water which can be released per month

   κ Energy per amount of water (megawatt hours per million cubic meters)

   In any given month, water may be spilled to respect the maximum reservior level. When water is spilled, it leaves the reservoir without producing energy.

(a) Formulate a *steady-state* monthly linear programming (LP) model which maximizes annual profit, taking market prices as given.

Use the following notation:

L_m Reservior level at the start of month m

R_m Water released during month m to generate electricity

S_m Water spilled during month m

(b) Write out key elements of the GAMS code for this model.

(c) Suppose that monthly demand as a function of price (p_m) is given by:

$$D_m = \alpha_m - \beta_m p_m$$

where p_m is the market price, and α_m and β_m are both positive constants. Formulate a *quadratic programming* (QP) model to determine the production profile which maximizes profit.

3. Let $n \in \mathbb{Z}$ with $n \geq 2$, and for all $1 \leq i < j \leq n$ consider the following sets of constraints:

$$x_i + x_j - y_{ij} \leq 1, \tag{2}$$

$$-x_i + y_{ij} \leq 0, \tag{3}$$

$$-x_j + y_{ij} \leq 0, \tag{4}$$

$$-y_{ij} \leq 0, \tag{5}$$

$$x_i \text{ integer}, \quad y_{ij} \text{ integer}, \tag{6}$$

We denote by QP^n_{LP} the polyhedron defined by $(2), \ldots, (5)$:

$$QP^n_{LP} = \{(x, y) \in \mathbb{R}^{n(n+1)/2} : (x, y) \text{ satisfies } (2), \ldots, (5)\},$$

and by QP^n the integer hull of QP^n_{LP}:

$$QP^n = \operatorname{conv}\{(x, y) \in \mathbb{R}^{n(n+1)/2} : (x, y) \text{ satisfies } (2), \ldots, (6)\}.$$

(a) Show that if $(x, y) \in QP^n_{LP}$, then $0 \leq x_i \leq 1$ and $y_{ij} \leq 1$ for all $1 \leq i < j \leq n$.

(b) What is the dimension of QP^n?

(c) Prove or disprove that $QP^2 = QP^2_{LP}$.

(d) Show that $QP^3 \neq QP^3_{LP}$ by giving a fractional vertex of QP^3_{LP}. Give a Gomory-Chvátal Rounding inequality that cuts off such fractional vertex.

4. Let $C = \{(x_1, x_2) : -x_1 + 2x_2 \le 0, -x_1 - 2x_2 \le 0\}$, i.e. $2|x_2| \le x_1$.

   (a) Define the normal cone to C at x (in the general case for a convex set C) and determine $N_C(x)$ for every $x \in \mathbf{R}^2$ in this specific example.

   (b) Consider the problem
   $$\min_{x\in C} \frac{1}{2}(x_1^2 - x_2^2) - px_1$$
   Show that for $p = 0$ the origin is a strict local minimizer of this problem. (If you use the second order sufficient conditions, be careful to define these precisely and define the sets that are used in its statement).

   (c) Now let p assume small positive values. How many stationary points (points satisfying the first order necessary conditions) are there near the origin? What are they? What kinds of points are they (local minimizers, saddle points, local maximizers)?

   (d) Suppose we change C to $\{(x_1, x_2) : 2x_2^2 \le x_1\}$. How does the answer to (b) change?

5. (a) Consider the following constrained optimization problem:
   $$\min_{x\in\mathbf{R}^n} f(x) \text{ subject to } c_i(x) = 0,\ i = 1, 2, \dots, m,\ \ h_j(x) \ge 0,\ j = 1, 2, \dots, r,$$
   where the functions f, c_i, $i = 1, 2, \dots, m$, and h_j, $j = 1, 2, \dots, r$ are all continuously differentiable. Write down KKT necessary conditions for optimality of a point x^*.

   (b) Write down the linear independence constraint qualification (LICQ) and Mangasarian-Fromovitz constraint qualification (MFCQ) for the problem in (a) at the point x^*.

   (c) Consider the Lagrange multipliers which, along with the point x^*, satisfies the KKT conditions for the problem in part (a). Denote these multipliers by λ_i^*, $i = 1, 2, \dots, m$ for the equality constraints and μ_j^*, $j = 1, 2, \dots, r$ for the inequality constraints. Show that when LICQ holds, the set of multipliers satisfying the KKT conditions contains a single point.

   (d) Consider the problem with inequality constraints only (that is, $m = 0$), and let μ_j^*, $j = 1, 2, \dots, r$ be optimal Lagrange multipliers for the inequality constraints, as in part (c). Show that when MFCQ holds, this set of multipliers is bounded. (Hint: Assume for contradiction that $\{\mu^k\}_{k=1,2,\dots} = \{(\mu_1^k, \mu_2^k, \dots, \mu_r^k)^T\}_{k=1,2,\dots}$ is a sequence such that each μ^k is a vector of optimal Lagrange multipliers for the inequality-constrained problem such that $\lim_k \|\mu^k\| = \infty$. Consider limit points $\bar{\mu}$ of the sequence of unit vectors $\{\mu^k/\|\mu^k\|\}$.)

   (e) Consider the following nonlinear program with complementarity constraint:
   $$\min_{x\in\mathbb{R}^n} f(x) \text{ subject to } g_1(x) \ge 0,\ g_2(x) \ge 0,\ g_1(x)g_2(x) = 0,$$
   where f, g_1, and g_2 are all continuously differentiable functions that map $\mathbb{R}^n$ to $\mathbb{R}$. Show that the LICQ and MFCQ cannot be satisfied at any feasible point of this problem.