Fall 2009 Qualifier Exam:
OPTIMIZATION

September, 2009

GENERAL INSTRUCTIONS:

1. Answer each question in a separate book.
2. Indicate on the cover of *each* book the area of the exam, your code number, and the question answered in that book. On *one* of your books list the numbers of *all* the questions answered. *Do not write your name on any answer book.*
3. Return all answer books in the folder provided. Additional answer books are available if needed.

SPECIFIC INSTRUCTIONS:

Answer 5 out of 8 questions.

POLICY ON MISPRINTS AND AMBIGUITIES:

The Exam Committee tries to proofread the exam as carefully as possible. Nevertheless, the exam sometimes contains misprints and ambiguities. If you are convinced a problem has been stated incorrectly, mention this to the proctor. If necessary, the proctor can contact a representative of the area to resolve problems during the *first hour* of the exam. In any case, you should indicate your interpretation of the problem in your written answer. Your interpretation should be such that the problem is nontrivial.

1. Consider the following linear program:

$$\begin{array}{lrcl} \min & 2x_1 - x_2 & & \\ & x_1 - x_2 & \geq & 4, \\ \text{s.t.} & 3x_1 + 2x_2 & \geq & 10, \\ & x_1, x_2 & \geq & 0. \end{array}$$

   (a) Write down the dual of this problem.

   (b) Find solutions for the primal and dual. (Hint: you could use tableaus labeled with both primal and dual variables.)

   (c) Suppose the right-hand side of the first constraint is changed from 4 to 10. Without performing any additional simplex iterations or referring to the tableau, give a lower bound on the optimal primal objective for the modified problem. Explain.

   (d) Does the solution of the primal problem change if you change the right-hand side of the second constraint to 12 (while leaving the right-hand side of the first constraint at 4)? Explain.

   (e) Suppose we modify (a) by changing the coefficient of x_1 in the objective from 2 to -2. What can you say about the properties of this modified problem and its dual?

2. (a) Suppose that the polyhedron

$$P(b) = \left\{ x \in \mathbb{R}^n_+ \mid Ax \leq b \right\},$$

   is bounded and non-empty $\forall b \in T \subset \mathbb{R}^m$. Consider the function $v : T \to \mathbb{R}$ defined as

$$v(b) \stackrel{\text{def}}{=} \max_{x \in P(b)} \{c^T x\}.$$

   What is the shape of $v(b)$? Prove your answer.

   (b) Prove that if the set

$$F = \{x \in \mathbb{R}^n_+ \mid Ax \leq b\}$$

   is unbounded, then there exists $k \in \{1, 2, \ldots n\}$ such that the linear program:

$$(LP)_k \qquad \max_{x \in F} x_k$$

has an unbounded optimal solution value.

Recall: A set T is unbounded if and only if $\exists z \in T$ such that $\|z\|_p \geq M \ \forall M \in \mathbb{R}$ (for any norm p).

3. I am a business person in Phoenix, Arizona with a need for an applied statistics solution, and was wondering if you could assist me.

   We are a Steelcase Office Environments distributor. Our sales by month are variable and have "peaks and valleys". We have our own employees, which when fully loaded for benefits and taxes, cost us $17.50 per hour. We have the option of using labor from an ourside agency at the cost of $25 per hour.

   What I am looking for is a model that will tell me the optimum base employee staffing level to maintain throughout the year, which can be added to from time to time with agency labor when our sales require additional staff over this employee base. Here are our projected sales and manhour needs by month for the year 2010:

MO.	SALES LEVEL	HOURS REQUIRED
JAN	3900	390
FEB	4200	420
MAR	3400	340
APRIL	3200	320
MAY	3100	310
JUNE	5900	590
JULY	3400	340
AUG	5800	580
SEPT	5500	550
OCT	3600	360
NOV	4200	420
DEC	6000	600

   (a) Can you provide me with the appropriate GAMS or AMPL model to use for this purpose?

   (b) What model would you use to minimize the expected cost if the hours required are instead +50 hours of the given value with probability 2/3, and -100 hours with probability 1/3?

(c) How would your proposed solution technique change if instead you were simply told that the hours required lie somewhere between the given value and +50 or -100 hours (without distribution information)?

4. Consider the following linear programming formulation of the $n \times n$ assignment problem: $\max_x \; a^T x \quad s.t. \quad Ax = e, x \geq 0$, where e is a vector of ones and $Ax = e$ represents the usual set of $2n$ assignment constraints (note: it is assumed that all n^2 arcs are present). For this problem suppose that there exists a non-optimal basic feasible solution (BFS) comprised of a numerical flow vector y and a set of basic arcs S.

   (a) Show that $y_{ij} = 1$ for all basic arcs ij with non-zero flow.

   (b) Prove that there exists a simple cycle flow with flow vector c and corresponding set of cycle arcs T with $|c_{ij}|$=1 for all arcs ij in T and such that $y + c$ is a feasible assignment with a better objective value.

   (c) Show that there is a set of arcs U such that $y + c$ and U comprise a BFS.

   (d) Can it be guaranteed that $y + c$ can be obtained from y in a single simplex pivot? Answer yes or no and explain your answer.

5. Consider the following *precedence-constrained* knapsack set:

$$X = \left\{ x \in \{0,1\}^n \;\middle|\; \sum_{i=1}^{n} a_i x_i \leq b, \; x_i \leq x_j, \; \forall (i,j) \in A \right\}$$

   where $a_i > 0$ for all $i \in N = \{1, \ldots, n\}$ and $A \subseteq N \times N$. For $i \in N$ define $D_i = \{j \in N \mid (i,j) \in A\}$ and for a set $C \subseteq N$, define $D(C) = C \cup \left(\cup_{i \in C} D_i\right)$.

   (a) A set $C \subseteq N$ is called an *induced cover* if $\sum_{j \in D(C)} a_j > b$. Prove that if C is an induced cover, then the inequality

$$\sum_{i \in C} x_i \leq |C| - 1$$

   is valid for X.

(b) Now consider the specific instantiation of this set

$$X = \left\{x \in \{0,1\}^4 \;\middle|\; 7x_1 + 5x_2 + 4x_3 + 4x_4 \le 12,\ x_2 \le x_4\right\}$$

and the inequalities

$$x_1 + x_2 + x_4 \le 2 \tag{1}$$
$$x_2 + x_3 \le 1 \tag{2}$$

which are valid for X (you may take this as given). For each of these inequalities, state whether or not it is facet-defining for the convex hull of X, and provide a proof of your answer. You may also take it as given that the convex hull of X is full-dimensional.

6. Suppose that f is a continuously differentiable function on $\mathbb{R}^n$. Write down (in concise form) first-order necessary conditions for the problem $\min_{x\in\Omega} f(x)$, where Ω is each of the following closed convex subsets of $\mathbb{R}^n$:

   (a) $\Omega = \mathbb{R}^n_+ := \{x \,|\, x_i \in [-1,1],\ i = 1,2,\ldots,n\}$.

   (b) $\Omega = \{x \,|\, Ax = b\}$ for some matrix $A \in \mathbb{R}^{m\times n}$ and vector $b \in \mathbb{R}^m$. (Assume that $\Omega \neq \emptyset$.)

   (c) $\Omega = \{\gamma v \,|\, \gamma \ge 0\}$, where v is a given vector in $\mathbb{R}^n$.

   (d) $\Omega = \{x \,|\, \|x\|_2 \le 1\}$.

   (e) $\Omega = \{x \,|\, \|x\|_1 \le 1\}$.

7. Let f be a closed proper convex function on $\mathbb{R}^n$. Suppose that there exist a real number β and a positive ϵ such that for each $z \in \mathbb{R}^n$ with $\|z\| < \epsilon$ we have for every x the inequality $f(x) \ge \langle z, x\rangle - \beta$.

   Must f then have a minimizer? Either prove that it does, or exhibit a counterexample.

8. Suppose $H \in \mathbb{R}^{n\times n}$ and $A \in \mathbb{R}^{m\times n}$. Prove the following are equivalent:

   (a) $Az = 0$ and $z \neq 0$ implies $\langle z, Hz\rangle > 0$

   (b) There exists $\bar{\gamma}$ such that $H + \bar{\gamma}A^TA$ is positive definite

Show if either of these holds, then $H + \gamma A^T A$ is positive definite for all $\gamma \geq \bar{\gamma}$.

Suppose now that H is symmetric and positive semidefinite on the null space of A, i.e. $Az = 0$ implies that $\langle z, Hz \rangle \geq 0$. Is it true that there is some $\bar{\gamma} > 0$ so that $H + \gamma A^T A$ is positive semidefinite for all $\gamma \geq \bar{\gamma}$.

Finally, give one or two sentences explaining why this result might be useful for $\min f(x)$ subject to $Ax = b$.