MATHEMATICAL PROGRAMMING

Instructions for Depth Exam Students: Answer any 6 of the following questions.

Instructions for Breadth Exam Students: Answer any 3 of the following questions.

1. Use the simplex method to solve the following problem:

$$
\begin{array}{ll}
\text{minimize} & 7\,|x| - y + 3z \\
\text{subject to} & \begin{array}{rcrcrcr}
 & & y & - & 2z & \leq & 4 \\
-x & - & y & + & 3z & \leq & -2 \\
3x & - & y & + & 3z & \geq & 0 \\
 & & y & \geq & 3, & z \leq & 1
\end{array}
\end{array}
$$

2. Let X be a convex set in the n–dimensional real space $\mathbb{R}^n$ and let f be a differentiable function on $\mathbb{R}^n$.

   (a) What can you say about the quantity

   $$\nabla f(\bar{x})(x - \bar{x})$$

   on X if $\bar{x}$ is a local solution of

   $$\underset{x \in X}{\text{minimize}} \quad f(x)$$

   (b) Let $\bar{x} \in \bar{X}$, where $\bar{X}$ is the solution set of $\min_{x \in X} f(x)$, that is

   $$\bar{X} := \arg\min_{x \in X} f(x)$$

If f is convex on $\mathbb{R}^n$, what is the relation between $\bar{X}$ and the set

$$\bar{Y} := \arg\min_{x \in X} \nabla f(\bar{x})(x - \bar{x})$$

Establish your claim.

3. Consider the problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g(x) \leq 0 \end{array}$$

where $f\colon \mathbb{R}^n \mapsto \mathbb{R}$ and $g\colon \mathbb{R}^n \mapsto \mathbb{R}^m$ are differentiable and convex functions on $\mathbb{R}^n$. Suppose that

$$x(\alpha) = \arg\min_{x}\{f(x) - \alpha \sum_{j=1}^{n} \log(-g_j(x)) \mid g(x) < 0\}$$

for $\alpha > 0$. What should α satisfy in order that

$$f(x(\alpha)) - \inf_{x}\{f(x) \mid g(x) \leq 0\} \leq 10^{-6}$$

Prove your claim.

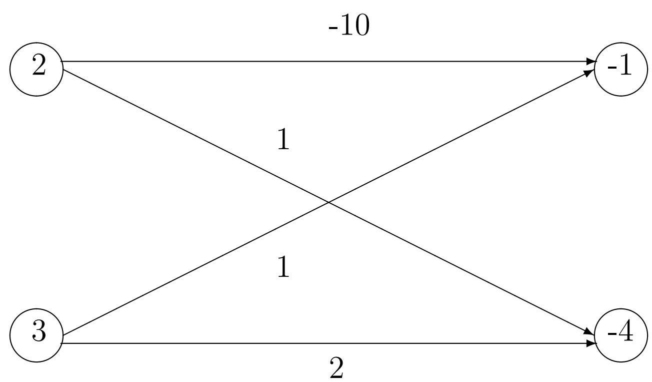
4. Consider the following problem.

   A company must assign n plants to n distinct locations. Each location will be assigned exactly 1 plant. The number of units to be shipped from p to q is $d_{pq} \geq 0$, $(p, q = 1, \ldots, n)$. The unit shipping cost from location i to location j is $c_{ij} > 0$, $(i, j = 1, \ldots, n)$. The plants are to be assigned to locations so as to minimize total shipping cost.

(a) Formulate the problem as a (not necessarily linear) integer program using only the n^2 variables x_{pi}, where $x_{pi} = 1$ implies that plant p is assigned to location i.

(b) Formulate the problem as a **linear** integer program using the variables of part (a) plus any additional variables that you require.

5.



(a) Formulate the "big M" problem corresponding to the min cost network problem above, where divergences are shown at the nodes and unit costs are shown on the arcs. Describe how you compute the **numerical** value of "big M" for this problem. Do one pivot of the network simplex method on the big M problem, indicating the primal and dual values before and after the pivot (start with the "all artificial" solution).

(b) Given the **orientation** of the arcs in the network above, what conclusion may be drawn regarding the **existence** of an optimal solution of the corresponding big M problem? Explain your answer.

(c) Given the result of part (b), what additional property (observable from the original data of this problem) relates the big M problem to the original problem? Explain your answer.

6. Let A be an integral matrix of full row rank. Show that all basis submatrices of A are unimodular iff for all integer vectors b

$$\{x \in \mathrm{I\!R}^n : \mathrm{Ax} = \mathrm{b},\ \mathrm{x} \geq 0\} = \mathrm{conv}\{\mathrm{x} \in \mathrm{I\!R}^n : \mathrm{Ax} = \mathrm{b},\ \mathrm{x} \geq 0, \mathrm{x\ integer}\}$$

Hint: *To show that a matrix B is unimodular it suffices to show that $B^{-1}t$ is integer for all vectors t.*

7. Consider the linear program:

$$\begin{array}{ll} \text{minimize} & \langle q, z\rangle \\ \text{subject to} & Mz + q \geq 0 \\ & z \geq 0 \end{array} \qquad (1)$$

where M is a skew-symmetric that is

$$M = -M^T.$$

Suppose z^* is a solution of (1). Give a solution of the dual of (1) in terms of z^*

8. Let f be a closed proper convex function from $\mathrm{I\!R}^n$ to $\mathrm{I\!R}$. Suppose that for each nonzero y in $\mathrm{I\!R}^n$ there exists some positive η such that $f(\eta y) > f(0)$.

(a) Prove that the set

$$L(1) = \{\, x \in \mathrm{I\!R}^{\mathrm{n}} \mid \mathrm{f(x)} \leq 1 \,\}$$

is compact.

(b) Can you draw any conclusion about the compactness of

$$L(n) = \{\, x \in \mathrm{I\!R}^{\mathrm{n}} \mid \mathrm{f(x)} \leq \mathrm{n} \,\},$$

where n is an arbitrarily large natural number? Explain.