MATHEMATICAL PROGRAMMING

Depth Exam: Answer any 6 of the following 8 questions

Breadth Exam: Answer any 3 of the following 8 questions

1. A textile firm is capable of producing 3 products in amounts $x_1,\ x_2,\ x_3$. Its production plan for the next month must satisfy the constraints:

$$x_1 + 2x_2 + 2x_3 \le 12$$
$$2x_1 + 4x_2 + x_3 \le f$$
$$x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0$$

The first constraint is determined by equipment availability and is fixed. The second constraint is determined by the availability of cotton, with f being the amount of cotton available. The net profits of the products are 2, 3 and 3 per unit respectively, excluding the cost of cotton.

(a) Find the optimal dual variable (shadow price) λ_2 of the cotton input as a function of f. Plot $\lambda_2(f)$ and the net profit $z(f)$, excluding the cost of cotton.

(b) The firm may purchase cotton on the open market at a price of $\frac{1}{6}$. However, it may acquire a limited amount s at a price of $\frac{1}{12}$ from a major supplier that it purchases from frequently. Determine the net profit of the firm $\Pi(s)$ as a function of s.

2. Consider the following linear system:

$$Ax = b$$
$$x \ge 0$$

where A is an $m \times n$ real matrix with rank $(A) = m$ and $0 \ne b \in \mathbb{R}^m$. Let $\Omega = \{x \in \mathbb{R}^n : Ax = b,\ x \ge 0\} \ne \emptyset$ and for each x let $X := \text{diag}\,(\mathrm{x_1}, \mathrm{x_2}, \ldots, \mathrm{x_n})$. Show that the two following statements are **equivalent**:

(a) rank $(AX) = m \qquad \forall x \in \Omega$

(b) b cannot be expressed as nonnegative linear combination of $m-1$ or fewer columns of A.

Hint: The matrix AX is comprised of positively-scaled columns of A and columns of zeros.

3. Let $P(x)$ denote the pure network flow problem

$$\begin{aligned} \min_x \quad & cx \\ \text{s.t.} \quad & Ax = b \\ & 0 \le x \le u, \end{aligned}$$

where A is a node-arc incidence matrix. Suppose that $\bar{x}$ is a BFS (basic feasible solution) of $P(x)$ and that x_1 and x_2 correspond to two pivot-eligible arcs (relative to $\bar{x}$).

(a) State conditions under which x_1 and x_2 can be "simultaneously" (i.e. in parallel) brought into the basis, producing the same new primal BFS that would result if they were brought in sequentially (in either order).

(b) State corresponding conditions for the dual variable updates associated with x_1 and x_2.

(c) Give a numerical example in which the conditions of part (a) are satisfied and the conditions of part (b) are violated.

4. Let $k(s)$ be a "separation counter" defined by

$$k(s) = \begin{cases} 0 & \text{if} \quad s < \delta \\ 1 & \text{if} \quad s \ge \delta \end{cases}$$

where δ is a given positive constant. Formulate as a mixed integer linear program the following pattern separation problem:

$$\begin{aligned} \max_{c,\alpha,s,t} \quad & \sum_{i=1}^{p} k(s_i) + \sum_{i=1}^{p} k(t_i) \\ \text{s.t.} \quad & cx_i - \alpha \ge \ \ s_i \quad (i = 1, \ldots, p) \\ & cy_i - \alpha \le -t_i \quad (i = 1, \ldots, p) \\ & \|c\|_\infty \le 1 \end{aligned}$$

where $x_1, \ldots, x_p$ and $y_1, \ldots, y_p$ are given sets of points in $\mathbb{R}^n$; and c (a row vector), α, $s = (s_1, \ldots, s_p)$, and $t = (t_1, \ldots, t_p)$ are unknowns. Be sure to define any constants (which may depend on the x_i and y_i) used in the formulation. (Note: Without loss of generality assume: $s_i \le \delta,\ t_i \le \delta, i = 1, \ldots, p$.)

5. Consider the problem $\min_{x \geq 0} f(x)$ where $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable and convex on $\mathbb{R}^n$. Assume that a solution $\bar{x}$ exists. For $z \in \mathbb{R}^n$ define $\big((z)_+\big)_i = \max\{z_i,\ 0\}, i = 1, \ldots, n$.

   (a) Suppose that for some $\hat{x} \geq 0,\ \nabla f(\hat{x}) > 0$. Find an upper bound on $\|\bar{x}\|_1$ in terms of $\hat{x}$ and $\nabla f(\hat{x})$, where $\|\cdot\|_1$ denotes the 1-norm.

   (b) Suppose, in addition, that f has a Lipschitz-continuous gradient, from which you can assume that for some number $L > 0$:

$$L\big\|y - x\big\|^2 \geq \big(\nabla f(y) - \nabla f(x)\big)(y - x) \geq \frac{1}{L}\big\|\nabla f(y) - \nabla f(x)\big\|^2$$

   where $\|\cdot\|$ denotes the 2-norm. Obtain for any $x \geq 0$ in $\mathbb{R}^n$, an upper bound on $\|\nabla f(x) - \nabla f(\bar{x})\|$ in terms of L, $\hat{x}$ and the quantities, $x\nabla f(x)$, $\big(-\nabla f(x)\big)_+$. (The last 2 quantities measure the violations by $x \geq 0$ of the Karush-Kuhn-Tucker conditions for the problem).

6. Consider the proximal point algorithm defined by

$$x^{k+1} = \arg\min_{x \in X} \big(f(x) + \frac{\gamma}{2}\|x - x^k\|^2\big)$$

where $\|\cdot\|$ denotes the 2-norm, $\gamma > 0$, f is differentiable and convex on $\mathbb{R}^n$, X is a convex subset of $\mathbb{R}^n$.

Define

$$\bar{X} := \arg\min_{x \in X} f(x) := \text{ set of minimizers of } f \text{ on } X$$

Suppose that for some k, $x^k \in \bar{X}$. Prove that $x^k = P(x^{k-1}\big|\bar{X})$ where $P(x\big|\bar{X}) = \arg\min_{y \in \bar{X}} \|x - y\|$.

Hint: You may want to use the fact that:

$$z = P(x\big|\bar{X}) \Leftrightarrow \langle x - z,\ y - z\rangle \leq 0 \quad \forall y \in \bar{X}$$

7. Let the function $f : \mathbb{R}^n \to \mathbb{R}$ have a Lipschitz continuous gradient on $\mathbb{R}^n$ with constant L. You are given a point $x \in \mathbb{R}^n$ and a direction vector $p \in \mathbb{R}^n$ such that $\nabla f(x)p < 0$ and $\|p\| = 1$, where $\|\cdot\|$ denotes the 2-norm.

   (a) For what interval of λ can you guarantee that $f(x+\lambda p) < f(x)$? Establish your claim.

   (b) What specific value of λ will give you the biggest guaranteed decrease in f? Establish your claim.

   (c) Suppose $p = -\nabla f(x)/\|\nabla f(x)\|$. What can you say about each accumulation point $\bar{x}$ of the sequence $\{x^i\}$ where $x^{i+1} = x^i + \lambda^i p^i$, and λ^i is chosen according to part (b)? Establish your claim assuming that $\nabla f(x^i) \neq 0$ for all i.

   Hint: Assume $f(x+\lambda p) - f(x) - \lambda \nabla f(x)p \leq \frac{L\lambda^2}{2} \|p\|^2$

8. Suppose f is a closed proper convex function on $\mathbb{R}^n$, and ρ is a fixed positive number. Let

$$f_\rho(x) = \inf_y g(x,y),$$

   where

$$g(x,y) = f(y) + (2\rho)^{-1}\|y-x\|^2.$$

   (a) Show that f_ρ is a convex function.

   (b) Show that the infimum in y of $g(x,y)$ is attained at a unique point of $\mathbb{R}^n$.

   Suggestion: As part of your answer for (b), consider establishing the following intermediate facts: (i) $g(x,\cdot)$ is lower semicontinuous; (ii) $g(x,\cdot)$ has bounded level sets; (iii) $g(x,\cdot)$ is strictly convex.