MATHEMATICAL PROGRAMMING

Depth Exam: Answer **6** questions, with at most **2** questions from **1, 2, 3.**
Breadth Exam: Answer **3** questions, with at most **2** questions from **1, 2, 3.**

1. Consider the following parametric LP:

$$\begin{array}{llcl}
\text{minimize} & 6x_1 + 5x_2 & & \\
\text{subject to} & x_1 - x_2 + x_3 & = & 1-\theta, \\
 & 10x_1 + 10x_2 - 10x_4 & = & 20, \\
 & 2x_1 + 3x_2 + x_5 & = & 18, \\
 & x \geq 0. & &
\end{array}$$

   For $\theta = 0$, an optimal basis consists of x_2, x_3, x_5. State the optimal solution and optimal value as a function of θ for $\theta \geq 0$.

   Note: An optimal solution may not exist for certain values of θ.

2. Suppose the linear program

$$\begin{array}{ll}
\underset{x}{\text{minimize}} & c^T x \\
\text{subject to} & Ax = b, \\
 & x \geq 0
\end{array} \qquad (1)$$

   has a solution and suppose that you have a good guess at an optimal solution x^0, where $x^0 \geq 0$ but $Ax^0 \neq b$. Using duality theory, show that for all sufficiently large positive γ,

$$\begin{array}{ll}
\underset{x,\lambda}{\text{minimize}} & c^T x + \gamma\lambda \\
\text{subject to} & Ax + (b - Ax^0)\lambda = b, \\
 & x, \lambda \geq 0
\end{array}$$

   has a solution, and that for any solution $(\hat{x}, \hat{\lambda})$, $\hat{x}$ is a solution of (1) and $\hat{\lambda} = 0$.

   Hint: Write the dual of the two linear programs above.

3. Solve the linear program:

$$\begin{array}{llcl}
\text{maximize} & x_1 & & \\
\text{subject to} & x_1 - x_2 & \geq & 0, \\
 & -x_1 - x_3 & \geq & -1, \\
 & -x_2 - 3x_3 & \geq & 0, \\
 & (x_1, x_2, x_3) \geq 0. & &
\end{array}$$

   Determine whether your solution is unique or not. Justify your answer. How about your dual optimal solution? Justify whether or not it is unique also.

4. Let $g : \mathbb{R}^n \mapsto \mathbb{R}^m$ be differentiable and convex on $\mathbb{R}^n$. Prove that the set

$$S := \{x : g(x) \leq 0\}$$

is empty if and only if there exists $(x, u) \in \mathbb{R}^n \mathrm{x} \mathbb{R}^m$ such that

$$ug(x) > 0, \quad u\nabla g(x) = 0, \quad 0 \leq u \leq e$$

where e is a vectors of ones in $\mathbb{R}^m$.

Hint: Consider $\min_{x \in \mathbb{R}^n} eg(x)_+$ where $(g(x)_+)_i = \max\{0, g_i(x)\}$.

5. In a modified Newton's method for solving $f(x) = 0$ ($f : \mathbb{R}^n \mapsto \mathbb{R}^n$), we iterate according to

$$x^{n+1} = x^n - [f'(x^1)]^{-1} f(x^n) \tag{2}$$

Assume that f is differentiable in a convex region D and that for some $x^1 \in D$, $[f'(x^1)]^{-1}$ exists. Assume that x^2 calculated according to (2) is in D, that

$$\rho = \left\| [f'(x^1)]^{-1} \right\| \sup_{x \in D} \left\| f'(x^1) - f'(x) \right\| < 1$$

and the ball

$$S = \{x | \left\| x - x^2 \right\| < \frac{\rho}{1 - \rho} \left\| x^1 - x^2 \right\| \}$$

is contained in D. Show that the modified method converges to a solution $x^* \in S$. Hint: You may wish to use contraction mappings and the mean value inequality

$$\|f(y) - f(z) - f'(x)(y - z)\| \leq \sup_{0 \leq t \leq 1} \|f'(z + t(y - z)) - f'(x)\| \, \|y - z\|$$

6. Let f be a closed convex function on $\mathbb{R}^n$. Suppose that there is a nonempty open subset Q of $\mathbb{R}^n$ such that f is finite on Q.

   (a) Show that f is proper.

   (b) Show that for each $x_0 \in Q$, the function (of the variable x^*) $f^*(x^*) - \langle x^*, x_0 \rangle$ has level sets that are all compact but not all empty.

   Suggestion: First suppose that x_0 were the origin, and see if you could solve the problem in that case. Then take care of the general case with a translation.

7. Consider the network transportation problem

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{i \in S} \sum_{j \in T} c_{ij} x_{ij} \\ \text{subject to} & \begin{array}{rcll} \displaystyle\sum_{j \in T} x_{ij} & \leq & a_i & \text{for all } i \in S \\ \displaystyle\sum_{i \in S} x_{ij} & \geq & b_j & \text{for all } j \in T \end{array} \\ & x_{ij} \geq 0 \end{array}$$

where S is the set of supply nodes and T is the set of demand nodes. Consider the aggregated problem obtained by aggregating all source nodes in a single node (the supply at this single node will be equal to the sum of the original individual supplies and the demand nodes will remain unchanged). Let $\hat{x}$ be an optimal solution for this aggregated problem.

(a) Using $\hat{x}$, construct a feasible solution for the original problem.

(b) Determine a lower bound on the optimal solution value for the original problem in terms of the optimal value (of the aggregated problem) and the reduced costs for the aggregated problem.

Note: You may assume $a_i > 0$, $b_j > 0$ and $c_{ij} \geq 0$ for all $i \in S$ and $j \in T$.

8. (a) Formulate the following as a **integer concave network flow** problem[1]:

   A KxL grid of cells is given, and each cell is to be assigned a processor index from the set $\{1, \ldots, P\}$; moreover, this assignment is to be **balanced** in the sense that each index in $\{1, \ldots, P\}$ is to be assigned to an equal number of cells (assume KL/P is an integer). Let $d_i(x)$ be the diversity measure for row i of an assignment x (the number of **distinct** processor indices in row i for the assignment x). We wish to minimize $\sum_{i=1}^{K} d_i(x)$ subject to the assignment and balancing constraints.

   (b) Prove that deletion of the integrality constraints (leaving only network constraints) has no effect on the optimal value.

9. Let

$$T := \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^{2n} : x \in \{0,1\}^n, \quad y \in \mathbb{R}^n_+, \quad \sum_{j=1}^{n} y_j \leq b, \quad y_j \leq a_j x_j \; \forall j = 1, \ldots, n \right\}$$

where b and a_j are nonnegative real numbers.

(a) Show that if C is a subset of $\{1, \ldots, n\}$ such that $\lambda = \sum_{j \in C} a_j - b > 0$ then

$$\sum_{j \in C} y_j \leq b - \sum_{j \in C} \max\{0, a_j - \lambda\}(1 - x_j) \tag{3}$$

is a valid inequality for T.

(b) Let now

$$S := \left\{ x \in \{0,1\}^n : \sum_{j=1}^{n} a_j x_j \leq b \right\}$$

[1] i.e., a problem with only network flow and integrality constraints, but with a concave objective function.

where, again, b and a_j are nonnegative real numbers and let C be a subset of $\{1, \ldots, n\}$ such that

$$\sum_{j \in C} a_j - b > 0 \text{ and } \sum_{\substack{j \in C \\ j \neq k}} a_j - b \leq 0 \quad \forall k \in C$$

Show that in this case, inequality (3) reduces to the well known valid constraint:

$$\sum_{j \in C} x_j \leq |C| - 1$$

where $|C|$ is the cardinality of C.
Hint: In part (b) define $y_j = a_j x_j$.