MATHEMATICAL PROGRAMMING

Depth Exam: Answer **6** questions, with at most **2** questions from **1, 2, 3.**
Breadth Exam: Answer **3** questions, with at most **2** questions from **1, 2, 3.**

1. Suppose we are solving the problem:

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array}$$

and we arrive at the following tableau:

x_1	x_2	x_3	x_4	-z	R.H.S.
5	1	0	a_1	0	b_1
-1	0	1	a_2	0	b_2
2	0	0	c	1	5

a. Identify the current basic solution.

b. Give conditions that ensure that the basic solution is a basic feasible solution.

c. Give conditions that ensure that the basic solution is an optimal basic feasible solution.

d. Give conditions that ensure that the basic solution is the unique optimal basic feasible solution.

e. Give conditions that ensure that there exists a class of solutions with objective values that are unbounded below.

f. Assuming that the conditions in part **e** hold, exhibit such a class of solutions.

g. Assuming that the conditions in part **b** hold, give all conditions under which you would perform a pivot on the element a_1.

2. Consider the LP:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \geq b, \quad x \geq 0 \end{array} \tag{1}$$

Suppose that a feasible solution $\bar{y}$ is given for the system:

$$\left[y^T A \leq c, \; y \geq 0 \right]$$

(a) Show that $\bar{y}$ may be used to determine a scalar M such that the following problem (where $e^T = (1, 1, ..., 1)$) is guaranteed to have an optimal solution:

$$\begin{array}{ll} \text{minimize} & c^T x + M e^T w \\ \text{subject to} & Ax + w \geq b, \quad x, w \geq 0 \end{array} \tag{2}$$

(Hint:Consider the dual of (2))

(b) Suppose that (x^*, w^*) is an optimal solution of the problem (2). Given the vectors x^* and w^*, what can be said about the original problem (1) ?

(Discuss the two cases $w^* = 0$ and $w^* \neq 0$.)

3. Consider the convex quadratic program

$$\begin{array}{ll} \text{minimize} & \frac{1}{2} x^T Q x + c^T x \\ \text{subject to} & Ax \geq b \end{array}$$

where $Q \in I\!R^{n \times n}$ is symmetric and positive semi-definite. Suppose the problem is feasible. Show that the following statements are equivalent.

(a) The objective function is bounded below on the feasible set.

(b) The implication holds:

$$[Av \geq 0, Qv = 0] \Rightarrow c^T v \geq 0$$

(c) There exist vectors r, s such that

$$c = Qr + A^T s, s \geq 0$$

4. Suppose $\bar{x}$ is a basic feasible solution for the network flow problem:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \quad 0 \leq x \leq u \end{array}$$

If (i,j) corresponds to an arc in the basis, show:

(a)The removal of (i,j) splits the tree corresponding to the basis into two trees, T_i and T_j.

(b) The value of $\bar{x}_{i,j}$ is given by the expression:

$\bar{x}_{i,j} = \sum_{v \varepsilon T_i} b_v - \sum_{k \varepsilon U_{i,j}} u_k + \sum_{k \varepsilon U_{j,i}} u_k,$

where $U_{i,j}$ is the set of non-basic arcs (r,s) at upper bound with $r \varepsilon T_i$ and $s \varepsilon T_j$, and $U_{j,i}$ is defined analogously. (Hint:Consider summing a subset of the constraints.)

5. Consider the maximum weight matching problem:

$$\begin{array}{ll} \text{maximize} & \displaystyle\sum_{(i,j)\in E} c_{ij}x_{ij} \\ \text{subject to} & \displaystyle\sum_{j:(i,j)\in E} x_{ij} \le 1 \quad \forall\, i \in N \\ & x_{ij} \in \{0,1\} \quad \forall (i,j) \in E \end{array}$$

where N is a set of nodes and $E \subseteq N \times N$ is a set of edges. All edge weights are positive. Let $H \subseteq E$ be a matching constructed by the following greedy algorithm:

> Choose edges of maximum weight such that each chosen edge does not meet any of the edges previously chosen.
> Stop when no more edges can be chosen.

The point

$$x_{ij} = \begin{cases} 1 & \text{if } (i,j) \in H \\ 0 & \text{otherwise} \end{cases}$$

is clearly feasible. Let

$$z^H = \sum_{(i,j)\in H} c_{ij}x_{ij}.$$

(a) Show that

$$z^H \le z^{**} \le 2z^H$$

where z^{**} is the optimal solution of the maximum weight matching problem. *Hint: Show that the point*

$$u_i^H = \begin{cases} c_{ij} & \textit{if } (i,j) \in H \\ 0 & \textit{otherwise} \end{cases}$$

is feasible for the dual of the LP relaxation.

(b) Construct a simple example to show that $z^H = \frac{1}{2}z^{**}$ is possible.

6. Consider the problem

$$\min_{x\ge 0} f(x)$$

where $f\colon R^n \to R$ is differentiable and convex on R^n.

(a) Write necessary and sufficient conditions, for $\bar{x}$ to be a solution of the problem, in terms of $\bar{x}$ and $\nabla f(\bar{x})$ only.

(b) Suppose in addition that f is strongly convex on R^n, that is, for some $k > 0$

$$(\nabla f(y) - \nabla f(x))\,(y - x) \ge k\|y - x\|^2 \qquad \forall x,\ y \in R^n$$

Derive the following error bound for any x in R^n :

$$\|x - \bar{x}\|^2 \le \alpha \cdot \|x\nabla f(x),\ \ (-\nabla f(x))_+,\ \ (-x)_+\|$$

where $\bar{x}$ is the solution, $\|\cdot\|$ is the 2-norm on R^{1+2n}, $(z_+)_i = \max\{z_i,\ 0\}$, $i = 1, \ldots, n$, and α is some positive constant that is independent of x.

Hint: α can depend on $\bar{x}$ and $\nabla f(\bar{x})$.

7. In a trust region algorithm for unconstrained optimization, the subproblems are of the form

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}s^T B s + g^T s \\ \text{subject to} & \|s\| \le \delta \end{array} \qquad (1)$$

where B is a symmetric positive definite matrix and $\delta > 0$. Let

$$s(\lambda) := -(B + \lambda I)^{-1} g$$

(a) Show that $\|s(\lambda)\|$ is a decreasing function of λ for $\lambda \ge 0$, provided $g \ne 0$.

(b) Show that (1) is solved by $s(\bar{\lambda})$ for the unique $\bar{\lambda} > 0$ such that $\|s(\bar{\lambda})\| = \delta$ unless $\|s(0)\| \le \delta$ in which case $s(0)$ solves (1).

(c) Suggest a practical scheme for determining a solution of (1).

8. Let f be a closed convex function on R^n, and let x_0 be a point in the relative interior of the effective domain of f. We know that in the special case in which f is differentiable at x_0, either $f'(x_0) = 0$ or else $-f'(x_0)$ is a descent direction for f at x_0.

In the present case, we do not assume f to be differentiable. We use the notation $f'(x; h)$ for the directional derivative of f at x in the direction h.

1. Explain the relationship between the function (of h) $f'(x_0; h)$ and the subdifferential $\partial f(x_0)$.

2. Show by example that if x_0^* is a nonzero element of $\partial f(x_0)$, it is not necessarily true that $f'(x_0; -x_0^*) < 0$.

3. Show that if d_0 is the projection of the origin onto $\partial f(x_0)$, then $f'(x_0; -d_0) = -\|d_0\|^2$. (Therefore, if $0 \notin \partial f(x_0)$ then $-d_0$ is a descent direction for f at x_0.)