MATHEMATICAL PROGRAMMING

Depth Exam: Answer 6 questions, with at most 2 questions from 1,2,3.
Breadth Exam: Answer 3 questions, with at most 2 questions from 1,2,3.

1. Consider the following LP:

$$\begin{array}{ll} \text{maximize} & 16x_1 + 24x_2 + 32x_3 \\ \text{subject to} & 3x_1 + 4x_2 + 6x_3 \le 200 \\ & 5x_1 + 6x_2 + 5x_3 \le 288 \\ & 8x_1 + 6x_2 + 5x_3 \le 400 \\ & x_1, x_2, x_3 \ge 0 \end{array}$$

The points

$$x^* = \begin{bmatrix} 0 \\ \frac{91}{2} \\ 3 \end{bmatrix} \qquad u^* = \begin{bmatrix} \frac{9}{2} \\ 1 \\ 0 \end{bmatrix}$$

are an optimal primal–dual solution of the above LP. Determine

(a) the range of values for the **coefficient of** x_1 in the objective (currently set at 16) such that x^* remains optimal;

(b) the range of values for the **rhs in the third** inequality (currently set at 400) such that u^* remains optimal.

2. Given an n-vector $\bar{x}$ and a set $Y := \{y \mid Ay \ge b\}$, formulate as a linear program the problem of finding a point in Y that minimizes $\|y - \bar{x}\|_1$ over all y in Y, where $\|t\|_1$ is the sum of the absolute values of the components of t. In addition, prove that the dual of this linear program is always feasible (even if Y is empty).

3. In a Newton method for solving $F(x) \ge 0$, $F\colon \mathbb{R}^n \to \mathbb{R}^m$, a direction from the point x is normally found from within $D(x)$, which is given by

$$D(x) := \{d \mid F(x) + F'(x)d \ge 0\},$$

where $F'(x)$ denotes the $m \times n$ Jacobian of F. In general, assuming that $D(x) \ne \emptyset$, there may be more than one direction in this set, so that the following is normally required in addition:

$$d \in D(x), \quad \|d\| \le \eta \, \mathrm{dist}\,(0 \mid D(x)),$$

where $\eta \ge 1$ and $\mathrm{dist}\,(x \mid D) := \inf_{y \in D} \|y - x\|$. By choosing a particular norm and a particular η, show one way to find such a d by solving either a single linear or a single quadratic program.

4. Consider the convex problem

$$\min \ f(x) \qquad s.t. \ Ax \le b$$

where $f\colon R^n \to R$ is differentiable and convex on R^n, A is an $m \times n$ matrix and $b \in R^m$. Suppose that for some $\hat{x}$ the Karush-Kuhn-Tucker conditions are **not** satisfied but $A\hat{x} = b$. Show how to use the solution of an appropriate linear program to determine a feasible $\bar{x}$ such that $f(\bar{x}) < f(\hat{x})$.

5. Suppose $\bar{x}$ is a solution of the monotone complementarity problem:

$$F(x) \ge 0, \ \ x \ge 0, \ \ xF(x) = 0 \qquad (NCP)$$

where $F : R^n \to R^n$ is monotone on R^n, that is

$$(y - x)(F(y) - F(x)) \ge 0 \qquad \forall y, \, x \in R^n$$

Give a precise condition which guarantees that the following function:

$$\theta(x) \ := xF(x) + ||(-F(x), \, -x)_+||_1 \ ,$$

has a global minimum at $\bar{x}$. Here $||z||_1$ denotes the 1-norm on R^{2n} and

$$(z_+)_i = \max \{z_i, \, 0\}, \ i = 1, \ldots, 2n, \ z \in R^{2n}.$$

6. Let $\alpha_1, \ldots, \alpha_n$ be a collection of nonzero real numbers, and consider the problem

$$\text{minimize} \sum_{i=1}^{n} e^{\alpha_i x_i}, \qquad \text{subject to} \sum_{i=1}^{n} \alpha_i^3 x_i = \beta, \qquad \sum_{i=1}^{n} \alpha_i^4 x_i = \gamma, \qquad (1)$$

where β and γ are given real numbers. Assume that (1) is feasible. Does it necessarily have an optimal solution? Be sure to specify any general theorems or other results that you use.

7. Consider the constraints $Ax = s, \ b \le x \le c$, where A is an $m \times n$ node-arc incidence matrix with corresponding flow vector x and s, b, c are constant vectors. Assume that these constraints have a feasible solution. Let $\bar{x}$ be a vector such that $b \le \bar{x} \le c$, but $g := s - A\bar{x} \ne 0$.

   (a) Prove that g has at least one positive component and at least one negative component.

   (b) Choose j such that $g_j > 0$ and define $T := \{(j,k) \mid \bar{x}_{jk} < c_{jk}\} \bigcup \{(k,j) \mid \bar{x}_{kj} > b_{kj}\}$. Prove that by adjusting $\bar{x}$ on T only, we may obtain a new flow vector x' such that $b \le x' \le c$ and $A_j x' = s_j$, where A_j is the the jth row of A.

8. Consider the following IP:

$$z_{IP} = \max \left\{ c^T x \,\middle|\, A^{(1)}x \leq b^{(1)}, A^{(2)}x \leq b^{(2)}, x \geq 0, \ x \text{ integer} \right\}$$

where all data are integer. Let

$$Q = \left\{ x \text{ integer} \,\middle|\, x \geq 0, \ A^{(2)}x \leq b^{(2)} \right\}$$

$$z_{LR}(\lambda) = \max \left\{ c^T x + \lambda^T \left(b^{(1)} - A^{(1)}x \right) \mid x \in Q \right\}$$

$$z_{LD} = \min \left\{ z_{LR}(\lambda) \mid \lambda \geq 0 \right\}$$

$$z_{LP} = \max \left\{ c^T x \,\middle|\, A^{(1)}x \leq b^{(1)}, x \in \text{conv}(Q) \right\}$$

(a) What is the relationship between z_{IP}, z_{LD} and z_{LP}?

(b) Suppose that for some positive scalars δ_1, δ_2 and some $\lambda \geq 0$

$$\lambda^T \left(b^{(1)} - A^{(1)}x^* \right) \leq \delta_1 \text{ and } c^T x^* + \lambda^T \left(b^{(1)} - A^{(1)}x^* \right) \geq z_{LR}(\lambda) - \delta_2$$

for some x^* feasible for the original integer problem. Show that

(i) $c^T x^* \geq z_{IP} - (\delta_1 + \delta_2)$,

(ii) x^* solves the original IP if $\delta_1 + \delta_2 < 1$.