Fall 1995 Qualifier Exam:
MATHEMATICAL PROGRAMMING

Instructions: Answer 5 of the following 7 questions.

1. Solve the following linear program

$$\begin{aligned} \max \quad & x_1 - x_2 + 7 \\ \text{s.t.} \quad & x_1 + 4x_2 = 5 \\ & x_1 - 3x_2 + x_3 = 7 \\ & x_1 + x_2 - x_3 \leq 8 \\ & x_1 \leq 1, \quad x_3 \geq 0 \end{aligned}$$

2. Show that any solvable linear program has a strictly complementary solution. (You may wish to consider the linear program

$$\begin{aligned} \max \quad & \epsilon \\ \text{s.t.} \quad & Ax \geq b \\ & c \geq A^T u \\ & b^T u \geq c^T x \\ & u + Ax - b \geq \epsilon 1_m \\ & x + c - A^T u \geq \epsilon 1_n \\ & x, u \geq 0 \end{aligned}$$

where 1_m and 1_n are appropriately dimensioned vectors of 1's.)

3. Consider the nonlinear program

$$\min_{x} \; f(x) \qquad \text{s.t. } g(x) \leq 0$$

where $f\colon R^n \to R$, $\quad g\colon R^n \to R^m$, are differentiable functions on R^n. Suppose that a linearization of the problem at some feasible point $\bar{x}$ has no *strict feasible descent direction*, that is the following linearized system has **no** solution in x:

$$f(\bar{x}) + \nabla f(\bar{x})(x - \bar{x}) < f(\bar{x})$$

$$g_I(\bar{x}) + \nabla g_I(\bar{x})(x - \bar{x}) < g_I(\bar{x}) = 0,$$

where $I := \{i | g_i(\bar{x}) = 0\}$.

(a) What optimality conditions are satisfied at $\bar{x}$? Justify.

(b) What additional assumptions on $\nabla g_I(\bar{x})$ can strengthen these optimality conditions? Justify.

Hint The Gordan Theorem may be useful in answering both (a) and (b).

4. Consider the nonlinear program $\min_{x \in X} \; f(x)$, where $f\colon R^n \to R$ and X is a subset of R^n. Define the exterior penalty function:

$$P(x, \alpha) := f(x) + \alpha\beta(x),$$

where $\beta\colon R^n \to R$ such that $\beta(x) = 0$ on X, else $\beta(x) > 0$. Let $\alpha_2 > \alpha_1 > 0$, and

$$x^i \in \arg\min_{x \in R^n} \; P(x, \alpha_i), \;\; i = 1, 2.$$

(a) Show that $f(x^i) \leq \inf_{x \in X} \; f(x)$.

(b) $f(x^2) \geq f(x^1)$.

(c) Suppose that you are given an arbitrary $x^0 \in X$, and that you have computed x^1 for some $\alpha_1 > 0$. How would you choose α_2 so that $\beta(x^2) < \delta$ for some given infeasibility tolerance $\delta > 0$?

5. Suppose f is a proper convex function on $\mathbf{R}^n$. Assume that for some nonzero $y^* \in \mathbf{R}^n$ and $\alpha^* \in \mathbf{R}$, and for all $x \in \mathbf{R}^n$, either $f(x) = +\infty$ or $\langle y^*, x \rangle \leq \alpha^*$. Let $x^* \in \mathrm{dom} f^*$ and consider the halfline from x^* in the direction of y^*. Exhibit the least value of α such that for each $t \geq 0$,

$$f^*(x^* + ty^*) \leq f^*(x^*) + t\alpha.$$

   You must prove that (i) the bound holds for your α, and (ii) your α is the best possible without more information on f. (You may find it useful to look at the support function of $\mathrm{dom} f$.)

6. Consider the nonlinear integer program:

$$(NIP) \qquad \min_x \quad 1 - x_1 x_2 \qquad \text{s.t.} \quad Ax \leq b, \quad x \geq 0, \quad x \text{ integer}$$

   a) Assuming that the constraints imply $x_1 \leq 1$ and $x_2 \leq 1$, formulate this problem as a *linear integer program* and explain why your formulation is an equivalent problem.

   Now suppose that $Ax \leq b$ is the single constraint $x_1 + x_2 \leq 1$.

   b) What is the optimal solution and optimal value of the corresponding nonlinear integer program?

   c) Consider an *arbitrary* linear integer program (not simply the one developed in part a)) that is equivalent to (NIP), assuming only that the constraints $Ax \leq b$ imply $x_1 \leq 1$ and $x_2 \leq 1$. Show that the LP relaxation of this arbitrary equivalent linear integer program will have optimal value not greater than $1/2$ when the constraints $Ax \leq b$ are specified to be $x_1 + x_2 \leq 1$. (Hint: use convexity properties of LP).

7. Consider a network flow problem with n nodes:

$$\min_x \quad cx \qquad \text{s.t.} \quad Ax = b, \quad x \geq 0,$$

where c is a vector all of whose components are 1 in absolute value.

Suppose the dual solution π (not necessarily dual feasible) satisfies the usual complementarity conditions with respect to a primal feasible solution defined on a spanning tree T.

a) If π_n=0, derive a bound for π_1 in terms of the problem data, and state necessary and sufficient conditions under which this bound will be attained.

b) If no assumptions are made on the value of any π_i, can a bound still be derived for π_1 from the complementarity conditions for this problem? Explain your answer.