MATHEMATICAL PROGRAMMING

Fall 1996 Qualifying Exam

Answer 5 questions from the following 7 questions.

1. (a) Prove or give a counterexample to the following: Among all the best L_1-approximations to a solution of

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \quad (i = 1, 2, \ldots, m) \tag{1}$$

there is always one that satisfies at least n of the m equations. (Hint: consider the case $m = n$.)

(b) Prove or give a counterexample to the following: Assume that (1) has no solution. Among all the best L_∞-approximations to a solution of (1), there is always one for which

$$\max_{i=1,2,\ldots,m} |b_i - \sum_{j=1}^{n} a_{ij}x_j| = b^T w$$

for some w satisfying $A^T w = 0$, $||w||_1 = 1$.

2. (a) Assume that $M \in R^{n\times n}$ and y satisfies $My + q \geq 0, y \geq 0$. Prove that the linearized LCP around y is solvable, that is:

$$\text{(LCPLP)} \qquad \arg\min_x \left\{ \left(y'(M + M') + q'\right)x \middle| Mx + q \geq 0, x \geq 0 \right\} \neq \phi$$

(b) If M is positive semidefinite, is the minimum value of the linearized problem positive? Explain.

3. Suppose

$$\bar{x} \in \arg\min \left\{ f(x) \middle| g(x) \leq 0 \right\}$$

where $f\colon R^n \to R$, $g\colon R^n \to R^m$ are differentiable convex functions on R^n and suppose that $g(\hat{x}) < 0$. Give an upper bound on the size of all optimal multipliers for this problem.

4. Consider the problem

$$(MP) \qquad\qquad \min \quad f(x) \quad \text{s.t.} \;\; g(x) \leq 0$$

where $f\colon R^n \to R$ and $g\colon R^n \to R^m$ are differentiable on R^n.

(a) Write the augmented Lagrangian $L(x, u, \alpha) : R^{n+m+1} \to R$ for this problem for $\alpha > 0$.

(b) How is an unconstrained stationary point of the augmented Lagrangian related to MP?

(c) Suppose f and g are convex on R^n and that for fixed α, u:

$$x(u, \alpha) \in \arg\min_{x\in R^n} L(x, u, \alpha)$$

Give a lower bound to the minimum of (MP) in terms of $\left(x(u, \alpha),\ u,\ \alpha\right)$.

5. Let C be a nonempty closed convex set in R^n and L a linear transformation from R^n to R^m. Let $x_0 \in R^m$, and suppose that x_0 does not belong to the set

$$X = \{x \mid x = Lc \text{ for some c} \in \text{C}\}.$$

Without further assumptions, can $\{x_0\}$ be *strongly* separated from X by a hyperplane? If your answer is yes, give a proof. If your answer is no, then (1) give a counterexample to the result without additional assumptions; (2) state the weakest additional condition that you can find under which the result is true; (3) give a proof under that condition. *Note:* Two nonempty convex sets U and V are strongly separated by a hyperplane H if for some positive ϵ, H separates the sets $U + \epsilon B$ and $V + \epsilon B$, where B is the unit ball.

6. The matrix rounding problem requires that we round the matrix elements (rounding in this context allows changing the value to either its floor or ceiling value), and round the row and column sums of the matrix so that the sum of the rounded elements in each row equals the rounded row sum and the sum of the rounded elements in each column equals the rounded column sum. Convert this problem into a max flow problem by introducing arcs (i, j') for each matrix element d_{ij}, an arc (s, i) for each row sum, an arc (j', t) for each column sum with appropriately chosen lower and upper bounds on all the arc flows (discuss how these bounds are chosen). Apply this conversion scheme to the following matrix rounding problem and solve the corresponding maximum flow problem:

				Row sum
	3.1	6.8	7.3	17.2
	9.6	2.4	0.7	12.7
	3.6	1.2	6.5	11.3
Column sum	16.3	10.4	14.5	

7. (a) Suppose that A is a set of r points in R^n and B is a set of s points in R^n. Assume that all coordinates of all points in A and B lie in the interval [0,1]. Formulate a mixed integer linear program for the problem of constructing a normalized classifying halfspace H such that the total number of misclassified points is as small as possible, where we define a misclassified point for a halfspace $H = \{x \mid cx \leq d\}$ as either (1) a point x of A such that $cx > d - 1$ or (2) a point y of B such that $cy < d + 1$. (Note: we say a halfspace is normalized if $\| (c, d) \|_\infty \leq 1$.)

   (b) Assuming that r=20, s=30, and n=4, give an estimate for the maximum number of nodes in a branch-and-bound tree for the MIP constructed above.