Spring 2008 Qualifier Exam:
OPTIMIZATION

February, 2008

GENERAL INSTRUCTIONS:

1. Answer each question in a separate book.
2. Indicate on the cover of *each* book the area of the exam, your code number, and the question answered in that book. On *one* of your books list the numbers of *all* the questions answered. *Do not write your name on any answer book.*
3. Return all answer books in the folder provided. Additional answer books are available if needed.

SPECIFIC INSTRUCTIONS:

Answer 5 out of 8 questions.

POLICY ON MISPRINTS AND AMBIGUITIES:

The Exam Committee tries to proofread the exam as carefully as possible. Nevertheless, the exam sometimes contains misprints and ambiguities. If you are convinced a problem has been stated incorrectly, mention this to the proctor. If necessary, the proctor can contact a representative of the area to resolve problems during the *first hour* of the exam. In any case, you should indicate your interpretation of the problem in your written answer. Your interpretation should be such that the problem is nontrivial.

1. Consider the following linear program:

$$\begin{aligned}
\max \quad & 2x_1 + x_2 \\
\text{subject to} \quad & x_3 = 5 - x_1 - x_2 \\
& x_4 = x_1 - x_2 \\
& x_5 = 21 - 6x_1 - 2x_2 \\
& x_1, x_2, x_3, x_4, x_5 \geq 0 \\
& x_1 \leq 3, x_2 \leq 2
\end{aligned}$$

Justify the following claim: the optimal values are obj=23/3, $x_1 = 17/6, x_2 = 2, x_3 = 1/6, x_4 = 5/6, x_5 = 0$. (Hint: re-express the problem using appropriate basic and non-basic sets and invoke appropriate optimality conditions. Note, however, that you may use any appropriate tools to do the justification and you are not required to use the mechanism proposed in the hint.)

2. Given a matrix $A \in \mathbb{R}^{m \times n}$ and a vector $b \in \mathbb{R}^m$, we define

$$X = \{x \in \mathbb{R}^n \mid Ax \leq b\}.$$

Suppose that X is nonempty. Given $\alpha \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$, prove that the property

$$X \cap \{x \mid \alpha^T x \leq \beta\} = X$$

(that is, $\alpha^T x \leq b$ is *redundant* for X) holds if, and only if, there exists a $y \in \mathbb{R}^m$ such that

$$A^T y = \alpha, \;\; y \geq 0, \;\;\text{ and } b^T y \leq \beta.$$

3. A farmer can lease land up to 1000 acres. She has to pay \$6 per acre (per year) if she leases up to 600 acres. For any land beyond 600 acres, she can lease at \$8 per acre. She grows corn on the land. She can grow corn at normal level or at an intense level (more fertilizer, frequent irrigation, etc.) Normal level yields 70 bushels/acre. Intense level yields 100 bushels/acre. The normal and intense levels require, respectively, 7 and 9 hours of labor per acre, and \$20 and \$30 in materials (such as seed, fertilizer, water, etc.) per acre. (On each acre, some amount can be at the normal level and some

at the intense level.) Harvesting requires 0.1 hours of labor per bushel harvested. Harvested corn can be sold up to any amount at the rate of $2.50 per bushel. The farmer can raise poultry at the same time. Poultry is measured in poultry units. To raise one poultry unit requires 24 bushels of corn and 20 hours of labor. She can either use the corn she herself has grown or buy corn from the retail market. She gets corn at the rate of $3.20/bushel from the retail market. She can sell at the price of $180 per poultry unit in the wholesale market up to 200 units. Any amount of poultry over 200 units sells for just $165 per unit. She and her commune can contribute 4,000 hours of labor per year at no cost. If she needs more labor, she can hire it at $4/hour up to 20,000 hours. Formulate a GAMS or AMPL model to maximize net profit. Briefly explain the computational advantages of your particular formulation.

4. Let N be a directed network corresponding to a complete graph on n nodes. Assume that a positive cost a_{ij} is given on each arc.

   (a) Formulate the problem of finding shortest paths from node 1 to each other node as a linear network flow problem. Be sure to state the divergence at each node.

   (b) Discuss how conformal decomposition may be applied to an optimal integer solution of the network flow problem of part (a) in order to obtain (n-1) shortest paths.

   (c) Give an example of a non-integer optimal solution for a network flow problem of the form given part (a) and explain what conformal decomposition will produce in that instance.

5. Consider the following sets:

$$
\begin{aligned}
A &= \{x \in \{0,1\}^n : 7x_1 + 5x_2 + 4x_3 + 3x_4 \geq 10\} \\
B &= \{x \in \mathbb{Z}^n_+ : 7x_1 + 5x_2 + 4x_3 + 3x_4 \geq 10\}
\end{aligned}
$$

   (a) Consider the inequality

$$x_1 + x_2 \geq 1. \tag{1}$$

   Is this a valid inequality for A? Is it a valid inequality for B? Justify your responses.

(b) Does the inequality from (a) define a facet of the convex hull of A? Does it define a facet of the convex hull of B? Justify your responses.

(c) Consider the inequality

$$x_2 + x_3 + x_4 \geq 1, \tag{2}$$

which is valid and facet-defining for

$$\begin{aligned} A_1 &= \{x \in A : x_1 = 1\} \\ &= \{x \in \{0,1\}^4 : 7x_1 + 5x_2 + 4x_3 + 3x_4 \geq 10, x_1 = 1\} \end{aligned}$$

What is the *maximum* value of α such that

$$\alpha x_1 + x_2 + x_3 + x_4 \geq 1 + \alpha$$

is a valid inequality for A? Does the inequality thus defined induce a facet of the convex hull of A? Justify your response.

6. Consider a line-search method for smooth unconstrained optimization that chooses the search direction p^k at iteration k to be a "partial steepest descent" direction, by selecting a subset of indices $\mathcal{S}_k \in \{1, 2, \ldots, n\}$ at each iteration, and defining

$$p_i^k = \begin{cases} -\left(\frac{\partial f}{\partial x_i}\Big|_{x=x^k} \right) & \text{if } i \in \mathcal{S}_k \\ 0 & \text{if } i \notin \mathcal{S}_k. \end{cases}$$

At all iterations k, we require $\mathcal{S}_k$ to contain the index j for which

$$\left| \left(\frac{\partial f}{\partial x_j}\Big|_{x=x^k} \right) \right|$$

is maximized over $j = 1, 2, \ldots, n$.

(a) If θ_k is the angle between p_k and $-\nabla f(x_k)$, show that $\cos\theta_k$ is uniformly bounded below by a positive constant.

(b) Suppose that the step length α_k at each iteration is chosen to satisfy the Wolfe conditions, and that f is bounded below and has a Lipschitz continuous gradient. Using the fact that

$$\sum_{k\geq 0} \cos^2\theta_k \|\nabla f(x^k)\|^2 < \infty,$$

show that any accumulation point of the sequence $\{x^k\}$ is a stationary point.

7. The following is a simplified mathematical model for the economical management of electrical power dispatch. There are n power generating plants indexed by $j = 1,\ldots,n$. Plant j is capable of producing any power amount x_j in a fixed interval $l_j \leq x_j \leq u_j$, where $0 \leq l_j < u_j < \infty$, the cost being $\phi_j(x_j) = c_j x_j + \frac{1}{2} b_j x_j^2$ with known coefficients $c_j > 0$ and $b_j > 0$. The plants are all connected to the same transmission network, so that when each produces an amount x_j the sum of these amounts enters the network; but due to transmission losses, which involve electrical interactions, the amount of power actually made available to customers is not this sum but

$$h(x) = h(x_1,\ldots,x_n) := \sum_{j=1}^{n} x_j - \frac{1}{2} \sum_{j=1,k=1}^{n,n} a_{jk} x_j x_k$$

for a certain symmetric, positive definite matrix $A \in \mathbf{R}^{n\times n}$ with entries $a_{jk} \geq 0$. It is assumed in the model that the entries a_{jk} are small enough that the partial derivatives $(\partial h/\partial x_j)(x)$ are positive at all vectors $x = (x_1,\ldots,x_n)$ having $l_j \leq x_j \leq u_j$. This ensures that h is an increasing function with respect to each variable over these ranges; in other words, an increase in power at one of the plants always results in an increase in power available from the network. Note that the highest value h can achieve is $h(u) = h(u_1,\ldots,u_n)$, whereas the lowest is $h(l) = h(l_1,\ldots,l_n)$. The exercise revolves around the following problem in these circumstances: for a given load demand d (power to be withdrawn from the network), with $h(l) < d < h(u)$, determine a scheduling vector $x = (x_1,\ldots,x_n)$ that meets this demand as cheaply as possible.

(a) Express this as a problem (P): $\min_{x\in X} f_0(x)$ subject to one equality or inequality constraint involving a scalar function f_1. Is this quadratic programming? convex programming? Is the corresponding Lagrangian

$L(x,y)$ convex in $x \in X$ for each $y \in Y$ (define Y yourself), as well as affine in $y \in Y$ for each $x \in X$?

(b) Does (P) have at least one optimal solution? At most one optimal solution?

(c) Show that, by virtue of the assumptions in the model, the standard constraint qualification (MFCQ) is satisfied at every feasible solution $\bar{x}$.

(d) If the Kuhn-Tucker conditions for (P) hold at $\bar{x}$, can you legitimately conclude that $\bar{x}$ is optimal? On the other hand, if they don't hold at $\bar{x}$, might $\bar{x}$ be optimal anyway?

(e) Show that the Kuhn-Tucker conditions come down to relations between the single Lagrange multiplier $\bar{y} = \bar{y}_1$ and the ratio of $\phi'_j(\bar{x}_j)$ to $(\partial h/\partial x_j)(\bar{x}_1, \ldots, \bar{x}_n)$ for $j = 1, \ldots, n$, and moreover that they imply $\bar{y}$ has to be positive.

8. We say that two subspaces of $\mathbb{R}^n$ are *independent* if their intersection is $\{0\}$.

   Let C_1 and C_2 be nonempty convex subsets of $\mathbb{R}^n$. Show that the following are equivalent:

   - Each element of $C_1 + C_2$ is expressible *in exactly one way* as a sum $c_1 + c_2$ of points $c_i \in C_i$, $i = 1, 2$.
   - The parallel subspaces $\operatorname{par} C_i$ of the sets C_i are independent.