Spring 2011 Qualifier Exam:
OPTIMIZATION

January 31, 2011

GENERAL INSTRUCTIONS:

1. Answer each question in a separate book.
2. Indicate on the cover of *each* book the area of the exam, your code number, and the question answered in that book. On *one* of your books list the numbers of *all* the questions answered. *Do not write your name on any answer book.*
3. Return all answer books in the folder provided. Additional answer books are available if needed.

SPECIFIC INSTRUCTIONS:

Answer 5 out of 8 questions.

POLICY ON MISPRINTS AND AMBIGUITIES:

The Exam Committee tries to proofread the exam as carefully as possible. Nevertheless, the exam sometimes contains misprints and ambiguities. If you are convinced a problem has been stated incorrectly, mention this to the proctor. If necessary, the proctor can contact a representative of the area to resolve problems during the *first hour* of the exam. In any case, you should indicate your interpretation of the problem in your written answer. Your interpretation should be such that the problem is nontrivial.

1. Consider the following problem:

$$\min_{x_1,x_2,x_3} \max(2x_1 - 3x_2 + x_3 + 2, -x_1 + x_2 + 2x_3 + 5)$$
$$\text{subject to} \quad x_1 - x_2 + x_3 = 4, \qquad x_1, x_2, x_3 \geq 0.$$

   (a) Formulate this problem as a linear program.

   (b) Solve this problem.

   (c) Find the dual of the LP formulation in (a).

   (d) Is the solution to the primal problem unique? If not, is the solution set bounded or unbounded?

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2. Suppose we have m non-empty polytopes

$$P_k = \{x \in \mathbb{R}^n \ : \ A_k x \leq b_k\} \tag{1}$$

for $k = 1, \ldots, m$. Here, A_k is $p_k \times n$ and $b_k \in \mathbb{R}^{p_k}$ for $k = 1, \ldots, m$. Write the optimization problem

$$\text{minimize } c^T x \ \text{ subject to } x \in \operatorname{conv}\left(\bigcup_{k=1}^{m} P_k\right)$$

as a linear program. Justify your formulation. Write down necessary and sufficient conditions on the cost vector c that guarantee that the optimal solution lies in P_1.

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3. In this problem, we will decide capital investments and an extraction sequence for a gold mine in Montana. This mining opportunity will take place for a limited number of time periods $T = \{1, 2, \ldots |T|\}$. Conceptually, the mine consists of an $m \times n$ array of blocks. (See Figure 1). Each block is defined by its vertical and horizontal position in the set $B = \{(i,j) \mid i \in \{1, 2, \ldots m\}, j \in \{1, 2, \ldots n\}$. Each block $(i,j) \in B$ has a volume v_{ij} and if mined will lead to a profit of p_{ij}. In order to mine the block at position (i,j), you must mine the blocks on top of it. Specifically, if $1 < j < n$, then in order to mine the block at position (i,j), the blocks at position $(i-1, j-1), (i-1, j)$, and $(i-1, j+1)$ must have been mined in a previous period. If $j = 1$, then in order to mine the block at position

i \ j				
	(1,1)	(1,2)	(1,3)	(1,4)
	(2,1)	(2,2)	(2,3)	(2,4)
	(3,1)	(3,2)	(3,3)	(3,4)

Figure 1: Example mine with $m = 3$ and $n = 4$. E.g., block $(2,2)$ cannot be excavated until after blocks $(1,1)$, $(1,2)$, and $(1,3)$ have been.

$(i,1)$, the blocks at positions $(i-1,1)$, and $(i-1,2)$ must have been mined in a previous period. Similarly, in order to mine the block at position (i,n), the blocks at positions $(i-1,n-1)$, and $(i-1,n)$ must have been mined in a previous period.

The mining is done by excavating machines. We must decide how many new excavators to lease in each period. Each excavator is able to clear a volume of Δ in each time period. The cost of leasing an excavating machine from period $t \in T$ until the end of the time horizon is ℓ_t. This (entire) cost is paid in period t and the machine is available until period $|T|$—i.e. you may not shorten the period of the lease.

We assume that the choice of excavating a block in a particular period is an all-or-nothing decision—either block (i,j) is completely excavated in period t, or no excavation is done on the block. We also assume that if a machine is newly leased in period t it is available for excavating immediately at the start of the period (and all periods thereafter).

The goal is to maximize the total cash on hand at the end of period $|T|$. We initially start with $\$K$. At the beginning of each period we can spend some of the currently available cash to lease additional excavating machines, but the cash balance is not allowed to go negative. Unspent cash accrues interest at a rate of δ/period—\$1 in period t will become $\$(1+\delta)$ in period $t+1$. At the end of the period we receive the profit from the blocks processed in that period.

(a) Write an *algebraic* description of the mine planning problem that achieves the objective and obeys all the problem restrictions. Be sure to clearly define your decision variables. State clearly assumptions that you made in your model that are not clarified in the problem description above.

(b) If time permits, demonstrate how your model would be implemented in either the GAMS or AMPL modeling language. You may assume that all sets and parameters will be instantiated (filled in with actual values) outside of the code you write.

4. Consider the following (general) two-stage stochastic program:

$$z^* \stackrel{\text{def}}{=} \min_{x \in X} \mathbb{E}_\omega \left[F(x, \omega) \right],$$

where $X \subseteq \mathbb{R}^n$, and

$$F(x, \omega) \stackrel{\text{def}}{=} \min_{y \in \mathcal{G}(x,\omega)} g(x, y, \omega)$$

is the optimal value of the second-stage problem for some function $g : \mathbb{R}^n \times \mathbb{R}^q \times \Omega \to \mathbb{R}$ and multifunction $\mathcal{G} : \mathbb{R}^n \times \Omega \to \mathbb{R}^m$. You may assume that $F(x, \omega)$ is "well-behaved". Specifically, $F(x, \cdot)$ is measurable, $-\infty < \mathbb{E}_\omega[F(x, \omega)] < +\infty \ \forall x \in X$, and $\min_{x \in X} F(x, \omega)$ exists for all $\omega \in \Omega$.

(a) Define the value

$$z_{\text{PI}} \stackrel{\text{def}}{=} \mathbb{E}_\omega \left[\min_{x \in X} F(x, \omega) \right]$$

to be the value of the *perfect information* solution. Prove: $z^* \geq z_{\text{PI}}$.

(b) Let $\omega_1, \ldots, \omega_N$ be an independent and identically distributed sample of ω (i.e., each ω_i has the same distribution as ω), and define

$$\hat{z}^N \stackrel{\text{def}}{=} \min_{x \in X} \frac{1}{N} \sum_{i=1}^{N} F(x, \omega_i).$$

Prove: $z^* \geq \mathbb{E}[\hat{z}^N]$, where here the expectation is taken with respect to the random sample $\omega_1, \ldots, \omega_N$.

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5. Consider the binary reverse knapsack set:

$$X = \left\{ x \in \{0, 1\}^n \mid \sum_{j=1}^{n} w_j x_j \geq b \right\}$$

where $b > 0$ and $0 < w_j \leq b$, for $j = 1, \ldots, n$.

(a) For any $S \subseteq \{1, \ldots, n\}$ define $b(S) = b - \sum_{j \in S} w_j$. Prove that if $b(S) \geq 0$, then the inequality

$$\sum_{j \notin S} \min\{w_j, b(S)\} x_j \geq b(S) \tag{2}$$

is valid for the set X. Hint: for an arbitrary $x \in X$, consider separately the case in which there exists $j' \notin S$ such that $x_{j'} = 1$ and $b(S) < w_{j'}$ and the case in which this does not hold.

(b) Consider the specific example:

$$X = \left\{ x \in \{0, 1\}^4 \mid 5x_1 + 4x_2 + 4x_3 + 3x_4 \geq 8 \right\}.$$

Specify inequality (2) for $S = \{2, 4\}$ and prove it is facet-defining for $\text{conv}(X)$. You may use the fact that $\text{conv}(X)$ is full-dimensional without proving it.

6. Let $b \in \mathbb{R}^n$. Let $\mathbf{1} \in \mathbb{R}^n$ denote the vector of all 1s. Compute an optimal solution of the one-dimensional unconstrained optimization problems

$$\text{minimize } \|x\mathbf{1} - b\|_p$$

for $p = 1, 2$ and ∞. In all cases, determine if the solution is unique and explain why or why not.

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7. Let $\{p_1, \ldots, p_K\}$ be a nonempty finite set of points in $\mathbb{R}^n$. A *geometric median* of the points p_k is a point $x \in \mathbb{R}^n$ that minimizes the sum $D(x) = \sum_{k=1}^K \|p_k - x\|$ of the Euclidean distances from x to the points.

Show that the minimum value of $D(x)$ is

$$\sup_{w_1,\ldots,w_K \in \mathbb{R}^n} \left\{ \sum_{k=1}^K \langle w_k, p_k \rangle \mid \sum_{k=1}^K w_k = 0,\ \|w_k\| \le 1 \text{ for } k = 1, \ldots, K \right\}.$$

Explain how you could calculate this value using an ordinary nonlinear-programming package that does not accept nonsmooth functions.

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8. Consider the following two problems, where b is a vector of nonpositive numbers, and μ and Δ are positive parameters:

$$\text{(A)} \qquad \min_x \ g^T x + \frac{\mu}{2} x^T x \ \text{ s.t. } \ Ax \ge b,$$

$$\text{(B)} \qquad \min_z \ g^T z \ \text{ s.t. } \ Az \ge b, \ \ \|z\|_2^2 \le \Delta^2.$$

(a) Write down KKT conditions for both problems.

(b) Explain why both (A) and (B) have a solution whenever μ and Δ are positive.

(c) Let $x(\mu)$ be the solution of (A) for some $\mu > 0$. Find a value of Δ such that $x(\mu)$ is also the solution of (B).

(d) For problem (A), show that $\|x(\mu)\|_2$ is a nonincreasing function of μ for $\mu > 0$.
