MATHEMATICAL PROGRAMMING

Depth Exam: Answer 6 questions, with at most 2 questions from 1, 2, 3.

Breadth Exam: Answer 3 questions, with at most 2 questions from 1, 2, 3.

1. Consider the following LP:

$$\min cx$$

s.t.
$$Ax = b, x \ge 0$$
.

- (a) Assuming that the feasible set is <u>bounded</u>, prove that if the optimal solution is <u>not unique</u>, then there exist <u>at least two</u> vertices of the feasible set that are optimal.
- (b) If the boundedness hypothesis is deleted in part (a), is the result still valid? Prove or provide a numerical counterexample.

2. Use the simplex method to solve the linear programming problem with lower and upper bounds:

3. Show that $\bar{x} = (1, 0, 1)$ is an optimal solution of the quadratic program

$$\max \quad \frac{1}{2}x^TQx + c^Tx$$
s.t.
$$Ax \le b$$

$$x \ge 0,$$

where

$$Q = \begin{bmatrix} -2 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} , \quad A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$
$$b = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{c}^{\mathrm{T}} = (2, 2, 2).$$

Be careful to quote any results that you use accurately.

4. Let (x^1, u^1) and (x^2, u^2) be distinct points in \mathbb{R}^{n+m} which satisfy the Karush-Kuhn-Tucker saddlepoint conditions for the problem

$$\min \ f(x) \quad \text{s.t.} \ g(x) \le 0$$

where $f: \mathbf{R}^n \to \mathbf{R}$ and $g: \mathbf{R}^n \to \mathbf{R}^m$.

- (a) What can you say about (x^1, u^2) and (x^2, u^1) ? Establish your claim.
- (b) If g is linear and f is differentiable, how are $\nabla f(x^1)$ and $\nabla f(x^2)$ related? Prove your claim.
- 5. Let $F: S \to S$ where S is a closed set in \mathbb{R}^n and let

$$||F(y) - F(x)|| \le \gamma ||y - x||, \ \forall x, y \in S \text{ and some } \gamma < 1.$$

- (a) Consider an algorithm $x^{i+1} = F(x^i)$, $\forall i \geq 0, x^0 \in S$. Prove that $\exists x^* \in S$ such that $x^i \to x^*$ and $x^* = F(x^*)$.
- (b) Consider the case where

$$F(x) = \frac{2x^3 + a}{3x^2}, \quad a > 0.$$

Prove that if x > 0, then $F(x) \ge \sqrt[3]{a}$. Using this fact to determine an appropriate S, show that the method converges from any starting point $x^0 > 0$ to the cube root of a.

- (c) Explain briefly why the method is locally quadratically convergent.
- 6. Consider the problem

min
$$f(x)$$
 s.t. $g(x) \le 0$

where $f: \mathbf{R}^n \to \mathbf{R}$ and $g: \mathbf{R}^n \to \mathbf{R}^m$.

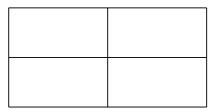
- (a) Write one complete step of the Augmented Lagrangian Algorithm.
- (b) Give a lower bound to $\inf \{f(x) | g(x) \le 0\}$ in terms of the current iterate of the Augmented Lagrangian Algorithm.
- (c) What can you say about a fixed point of the Augmented Lagrangian Algorithm?
- 7. Suppose that f and g_1, \ldots, g_m are closed proper convex functions from \mathbf{R}^n to \mathbf{R} , all having the same effective domain D. Consider the parametrized optimization problem

inf
$$f(x) + \langle u, x \rangle$$
, subject to $g_i(x) \le v_i \ (i = 1, ..., m)$, (P(u, v))

where u and v are prescribed vectors in \mathbf{R}^n and \mathbf{R}^m respectively.

For any particular choice of functions f and g_1, \ldots, g_m as above, do there always exist values \hat{u} and \hat{v} of the parameters such that both the problem $P(\hat{u}, \hat{v})$ and its dual with respect to the standard right-hand side parametrization have optimal solutions? Justify your answer.

8.	Two machines M_1 and M_2 are available for at most 8 and 10 hours respectively
	for the production of two products A and B , that may be sold for \$10 and \$11
	per unit respectively. The production rates (units per hour) are given in the
	following table:



Sales are limited to at most 20 units of A and 50 units of B.

- (a) Formulate the problem of maximizing total revenue as a <u>generalized</u> <u>network flow</u> problem. Give both an <u>algebraic</u> and a <u>graphical</u> formulation.
- (b) If a parallel computer is available, describe how, starting with an <u>all-slack</u> basis, two pivots may be performed in parallel, producing an improved basic feasible solution. Show the flows for this basic feasible solution graphically.