MATHEMATICAL PROGRAMMING

Spring 1998 Qualifying Exam
February 2, 1998
Instructions: Answer 5 of the following 7 questions.

1. Glassco manufactures wine glasses, beer glasses, champagne glasses and whiskey glasses. Each type of glass uses time in the molding shop, time in the packaging shop, and a certain amount of glass. The resources required to make each type of glass are given in the following table:

	x_1 WINE GLASS	x_2 BEER GLASS	x_3 CHMPGNE GLASS	x_4 WHISKEY GLASS
Molding time	4 minutes	9 minutes	7 minutes	10 minutes
Packaging time	1 minute	1 minute	3 minutes	40 minutes
Glass	3 oz	4 oz	2 oz	1 oz
Selling price	\$6	\$10	\$9	\$20

At present, 600 minutes of molding time, 400 minutes of packaging time and 500 oz of glass are available.

(a) Write down the LP that Glassco should solve, assuming the company wishes to maximize revenue.

(b) Write down the dual of this LP problem.

(c) Glassco claims that the optimal solution to the (primal) problem is $z = 2800/3$, $x_1 = 400/3$, $x_2 = 0$, $x_3 = 0$ and $x_4 = 20/3$. Is this the case? Justify, and quote any relevant theory you use accurately.

2. (i) Set up a single linear program to find a "largest" element in the set:

$$X := \{x \mid Ax \geq b,\ x \geq 0\},$$

where A is an $m \times n$ matrix and b is an $m \times 1$ vector, using some norm $\|x\|$ on $\mathbb{R}^n$ that makes this possible.

(ii) Set up a finite number (not exponential in n) of linear programs to do the same for the set:

$$Y := \{y \mid Ay \geq b\},$$

using some norm $\|y\|$ on $\mathbb{R}^n$ that makes this possible.

3. Let

$$x(A,a) \in \arg\min f(x) \;\; s.t. \;\; Ax \le a, \; x \ge 0,$$

$$x(B,b) \in \arg\min f(x) \;\; s.t. \;\; Bx \le b, \; x \ge 0,$$

where $A, \; B$ are $m \times n$ matrices, $a, \; b$ are $m \times 1$ vectors, $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a differentiable convex function on $\mathbb{R}^n$, and

$$B \ge A, \;\; \text{and} \;\; a \ge b,$$

where $B \ge A$ means that B is componentwise greater than or equal to A. Relate $f(x(A,a))$ and $f(x(B,b))$ by an inequality and justify.

4. For the problem

$$\min f(x) \text{ subject to } g(x) \le 0$$

where $f\colon \mathbf{R}^n \to \mathbf{R}$ and $g\colon \mathbf{R}^n \to \mathbf{R}^m$, consider the Absolute Value penalty

$$p_a(g(x)) = \|g(x)_+\|_1,$$

and the Courant penalty

$$p_c(g(x)) = \|g(x)_+\|_2^2,$$

where for $y \in \mathbf{R}^m$, y_+ denotes $(y_+)_i = \max\,\{y_i, 0\}, \;\; i = 1, \ldots m.$ Consider now the problem

$$P1 : \min x^3 - x \text{ subject to } 0 \le x \le 1$$

(a) Sketch the Absolute value and Courant penalty terms for P1.

(b) For each positive integer k, compute the minimizer x_k of

$$P_k(x) = f(x) + kp_c(g(x))$$

for the f and g defined in P1.

(c) For each positive integer k, compute the minimizer x_k of

$$F_k(x) = f(x) + kp_a(g(x))$$

for the f and g defined in P1.

(d) Use P_k to solve

$$P2 : \min x_1 + x_2 \text{ subject to } x_1^2 - x_2 \le 2$$

(e) Show that F_k has no stationary points off the parabola

$$x_1^2 - x_2 = 2$$

for problem P2 with $k > 1$ and compute the minimizer of $F_k(x)$.

5. For $i = 1, \ldots, m$ let a_i be elements of $\mathbb{R}^n$, let α_i be scalars, and for $x \in \mathbb{R}^n$ define $f_i(x)$ to be $\langle a_i, x \rangle - \alpha_i$. Let $f(x) = \max_i f_i$.

   (a) Show that f is a closed proper convex function that is finite on all of $\mathbb{R}^n$.

   (b) Given $x^* \in \mathbb{R}^n$, show how one can compute the conjugate function $f^*(x^*)$ by using linear programming.

   (c) Determine the effective domain of f^* (Explain geometrically how to find it given the data a_i and α_i. Does it depend on the a_i? On the α_i?)

   (d) Suppose x_0 is a point of $\mathbb{R}^n$. Explain how to find the set of "weak descent" directions z having the property that for each $\lambda \geq 0$, $f(x_0 + \lambda z) \leq f(x_0)$. Determine, in particular, whether or not this set depends on x_0 and, if so, how.

   (e) For the case in which $n = 2$, $x_0 = (1, 2)$, the a_i are $(-2, -1)$, $(-5, -5)$, and $(1, -3)$, and the α_i are all equal to 1, determine the set of weak descent directions.

   Justify your answers.

6. Construct a linear mixed-integer program to model the following "avoid 1" problem:

$$min \quad \sum_{i=1}^{n} f(x_i) \qquad s.t. \quad Ax = b, \quad 0 \leq x \leq 2e,$$

   where x is an n-vector, e is a vector of 1's, and $f(x_i) = 2 - |1 - x_i|$.

   "Check" your model by demonstrating that it yields the correct objective value at $x = 2e$ if $2Ae = b$. What is this value?

7. Consider a network with n nodes and a set of values d_{ij} assigned to the arcs that give the "distances" between all pairs of distinct nodes i and j.

   (a) Assuming all $d_{ij} > 0$, and using a node-arc incidence matrix, construct a single linear program that represents the problem of determining a set of shortest paths from node 1 to all other nodes. (Be sure to specify the RHS of the LP.) Discuss how the shortest paths may be obtained from an appropriate optimal solution of the LP.

   (b) Construct a numerical example with all $d_{ij} < 0$ in which the algebraic LP constructed in part (a) fails to correctly represent the shortest path problem and demonstrate the non-equivalence of the two problems by stating a feasible solution of the LP whose objective value is less than the optimal value of the shortest path problem.

   Note: In both parts (a) and (b) above, "paths" are assumed to be simple paths in which no arc appears more than once.