Numerical Analysis
Depth Examination
Fall 02

You should answer five out of the eight problems below.

·• **1** •· Let A be a real symmetric matrix, and let $[a\,..\,b]$ be an interval.

(a) Describe an algorithm that determines, in finitely many steps, the number of eigenvalues (counting multiplicities) that A has in the interval $[a\,..\,b]$. Be sure to state basic theorems from Linear Algebra that, in your opinion, guarantee the success of your algorithm.

(b) Apply the algorithm you proposed under (a) to determine the number of eigenvalues in the interval $[-6\,..\,0]$ of the matrix

$$\begin{bmatrix} -9 & 3 & -3 \\ 3 & 2 & -5 \\ -3 & -5 & 10 \end{bmatrix}.$$

·• **2** •· This question deals with numerical quadrature, but no connection between its three parts is intended.

(a) Use extrapolation to the limit to derive Simpson's rule from the Trapezoid rule and the Midpoint rule.

(b) Which quadrature rule for $\int_a^b f(x)\,\mathrm{d}x$ do you obtain by the following process? You approximate that integral by $\int_a^b p_2(x)\,\mathrm{d}x$, with p_2 the unique polynomial of degree < 3 that agrees with f at the three points a, $(a+b)/2$, and b.

(c) Discuss in some (qualitative) detail how you would go about obtaining, in an efficient way, an accurate value for

$$\int_{-1}^{1} \frac{1+|x^2-1/4|}{\sqrt{1-x^2}}\,\mathrm{d}x.$$

Justify your approach, by citing relevant facts about the quadrature rule(s) you are proposing to use.

·• **3** •· This question deals with best approximation by Π_n (polynomials of degree $\le n$) to continuous functions in the uniform (=max) norm.

(a) Given a continuous f on the interval $[a\,..\,b]$, state and prove a condition on $p \in \Pi_n$ that guarantees it to be the best approximation to f. Your condition should be in terms of the oscillatory behavior of the error $f - p$.

(b) Determine a $b = b(n)$ (as small as you can make it) such that the best approximation from Π_n to $f : x \mapsto \sin x$ on the interval $[0\,..\,b(n)]$ is the zero function.

(c) Weierstraß' Theorem asserts that every continuous function can be approximated arbitrarily well by polynomials. How could such an assertion co-exist with the observation made in (b) above?

·• **4** •·

(a) Define what it means for a linear map, from some normed linear space to some normed linear space, to be *bounded below*, and relate it to the property of the map having a bounded inverse.

(b) Give an example of a bounded linear map that is bounded below yet not invertible.

(c) Give an example of a bounded linear map that is invertible yet not bounded below.
(d) State the basic theorem in Functional Analysis that implies that a bounded linear map, from some Banach space to some Banach space, is boundedly invertible if and only if it is 1-1 and onto.

·**• 5 •**· Consider the Initial-Value Problem:

$$y' = \frac{1-y^2}{1+y^2}, \quad y(0) = 0, 0 \le t \le 2.$$

Consider also the numerical method for solving it:

$$Y_{k+2} = Y_k + 2h\frac{1-Y_{k+1}^2}{1+Y_{k+1}^2}, \quad (k+2)h \le 2,$$

$$Y_0 = 0, \quad Y_1 = h.$$

Discuss the behaviour of the solution $(Y_k)_k$ along the following lines:
(a) Convergence to the true solution?
(b) Order of convergence?
(c) Stability of the process?

·**• 6 •**· Of all the finite difference schemes for $u_t + au_x = 0$ of the form

$$Au_{m+1}^{n+1} + Bu_m^{n+1} + Cu_{m+1}^n + Du_m^n = 0$$

show that the second-order accurate box scheme is the most accurate.

·**• 7 •**· Consider both (a) a multistep scheme for $u_t + au_x = 0$ and (b) a multistep scheme for $u_{tt} + a^2u_{xx} = 0$. The stability condition for (a) is that all roots must be at most 1 in magnitude and if they are 1 in magnitude, then they must be simple roots. For (b) the stability condition is similar except that it allows double roots on the unit circle. Discuss why there is this difference between these two stability conditions.

·**• 8 •**· This question deals with the discrete five-point Laplacian $\nabla_h^2 u = 0$ on the unit square with boundary data specified on the boundary.

(a) Consider solving for the solution by the Jacobi method. Show that if $u_{\ell,m}^0 \le v_{\ell,m}^0$ for all grid indices (ℓ, m) at iteration 0, then $u_{\ell,m}^\nu \le v_{\ell,m}^\nu$ at all grid indices (ℓ, m) and all iterations ν.

(b) Use the result in (a) to prove that a solution exists to $\nabla_h^2 u = 0$ with u equal to the data on the boundary.