Numerical Analysis

1. A smooth function f is approximated by a polynomial $p \in \Pi_n$ (i.e., a polynomial of degree $\leq n$). The error function $f - p$ is known to vanish at the $n+1$ equidistant points

$$-1, \quad -1 + \frac{2}{n}, \quad -1 + \frac{4}{n}, \quad \ldots, \quad 1.$$

(a) Give your best estimate for the pointwise error

$$|f(x) - p(x)|, \quad x \in [-1, 1].$$

(Note: your bound should depend on x.)

(b) Derive the bound you had provided in (a).

(c) Estimate $\|f - p\|_{\infty,[-1,1]}$, for $f(x) = e^x$, and $n = 5$.

2. **(a)** Find the best uniform approximation from Π_4 to the function $f(x) = x^5$ on the interval $[-1, 1]$.

(b) In numerical integration, symmetric rules are preferred. Write a formula for the error of the rule

$$\int_a^b f(t)\, dt \approx \sum_{j=0}^{2n} w_j f(x_j),$$

exact for all polynomials in Π_{2n}, and explain, in terms of that formula, the reason for the above preference.

3. In order to find a solution in the interval $[a, b]$ for the equation $x = g(x)$, fixed point iteration

$$x^{n+1} = g(x^n), \quad x^0 = (a+b)/2$$

is employed.

(a) Prove or disprove: if $g'(x) > 1$ on $[a, b]$, then there is no solution to the equation.

(b) Prove or disprove: if $g'(x) > 1$ on $[a, b]$, the iterations must diverge.

(c) Explain why you agree/disagree with the following statement: "If $3/4 \geq |g'(x)| \geq 2/3$ on $[a, b]$, then the bisection method will converge to the solution faster than fixed point iteration, and hence should be preferred."

4. **(a)** Describe the Gauss-Seidel iterative method for the solution of a linear algebraic system $A? = b$. Include in your description at least one sufficient condition for convergence.

(b) Prove: if A is upper triangular, then Gauss-Seidel iteration converges.

5. Let λ be an eigenvalue of the matrix

$$A := \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

with associated (nontrivial) eigenvector $\mathbf{x}$.

Prove: If also $\mathbf{y}$ is an eigenvector of A corresponding to the eigenvalue λ, then $\mathbf{y} = c\mathbf{x}$ for some scalar c.

6. **(a)** Supply a definition for $\omega_f(h)$, the uniform modulus of continuity of the map f from some metric space to another.

(b) Describe how you would use ω_f in order to determine the size of the permissible error δ in the argument x if $f(x + \delta)$ is to equal $f(x)$ within a prescribed tolerance ϵ.

(c) Give as much information as you can about ω_f in case f is a proper contraction from some normed linear space X to some normed linear space Y.

7. You are given the initial value problem

$$y' = f(y) := \frac{y}{1+y^2} \qquad \text{on } [0,10]$$
$$y(0) = 1/2$$

You are asked to solve it numerically via the method

$$Y_{k+1} = Y_{k-1} + (2h)f(Y_k) \qquad k = 1, 2, \ldots$$
$$Y_0 = 1/2, \quad Y_1 = Y_0 + hf(Y_0)$$

Discuss the behavior of the error. Specifically, give a detailed answer to the following two questions:

(a) Is the method convergent?

(b) Assuming the answer to (a) is positive, is there anything special about the behavior of the error?

8. Show that any finite difference scheme of the form

$$v_m^n = \sum_{j=0}^{L} a_j v_{m+j}^n$$

that is consistent with

$$u_t + u_x = 0$$

is unstable.

9. Consider the Crank-Nicolson scheme applied to $u_t = u_{xx}$ for $x \in \mathbb{R}$ and $t > 0$. Determine conditions on $\Delta t/(\Delta x)^2$ such that the solution v_m^n satisfies

$$\max_m |v_m^n| \to 0 \quad \text{as} \quad n \to \infty$$

for all initial data v^0 with $\max_m |v_m^0|$ finite.

10. **(a)** Let λ be a nontrivial linear functional on some linear space X. Prove: For all $x \in X$ and all $y \in X \setminus \ker \lambda$, the modified element $x - (\lambda x/\lambda y)y$ is in $\ker \lambda$.

(b) Relate the abstract construction in (a) to Gauss elimination.

(c) Relate the abstract construction in (a) to Gram-Schmidt orthogonalization.

(d) Describe in some detail how Gauss elimination would look if it were carried out in the same order as Gram-Schmidt orthogonalization.