Fall 2003 Qualifier Exam: OPTIMIZATION

September, 2003
Instructions: Answer 5 out of 8 questions

1. Consider the following linear program:

$$\begin{array}{lrcl} \min & 8x_1 - x_2 & & \\ \text{subject to} & x_1 - 2x_2 & \geq & -2, \\ & x_1 - x_2 & \geq & -7, \\ & x_1, x_2 & \geq & 0. \end{array}$$

   (a) Solve this problem.

   (b) Write down the dual of the given problem and the KKT conditions.

   (c) Find a dual solution u^* (by inspection of the KKT conditions).

   (d) Suppose that the objective function is replaced by the following quadratic:

$$ax_1^2 + bx_2^2 + 8x_1 - x_2,$$

   where a and b are nonnegative parameters. Write down the modified KKT conditions for the resulting problem.

   (e) How large can we make a and b before the solution of the quadratic problem becomes different from the solution of the original linear program?

2. Consider the function

$$f_d(t) = \min_{x \in \mathbf{R}^n} c'x \;\text{ s.t. } Ax = b + td,\; x \geq 0,$$

   where A is an $m \times n$ matrix, and assume that for $t = 0$ the corresponding LP has a unique optimal solution having m positive variables.

   (a) Show that for any m-vector d , there exists an $\alpha(d) > 0$ such that $f_d(t)$ is linear on $[0, \alpha(d)]$.

   (b) State an expression giving f_d as a linear function of t on that interval.

   (c) Give a counter-example showing that the claim in (a) can be false if the optimal solution is not unique. (Note that the stated assumptions do not include linear independence.)

3. New York City has 10 trash districts and is trying to determine which of the districts should be sites for trash processing plants. It costs \$1000 (annually) to haul one ton of trash one mile. The central location of each district, the number of tons of trash produced per year by each district, the annual fixed cost (in millions of dollars) of operating a trash processing facility in that district, and the variable cost (in dollars per ton of trash) for processing trash in that district are shown in the table.

   For example, district 3 is located at coordinates (10,8). This district produces 555 tons of trash per year. If we were to operate a processing plant in district 3, we would incur a fixed cost of \$1,000,000, plus \$51 for each ton of trash processed. District 3's own trash could be processed at this plant. If district 3's plant is operated, trash from other districts could be shipped to district 3 for processing, at added cost for transportation. For example, trash from district 2 would incur a shipping cost of \$1000 for each of its 874 tons of trash for each of the $\sqrt{(10-2)^2+(8-5)^2} \approx 8.54$ miles separating districts 2 and 3. (Of course, it would also incur trash processing costs at district 3.)

   Each plant can handle at most 1500 tons of trash. Each district must send all its trash to a single site.

   Write a model whose solution indicates what trash processing plants should be used, and the plants to which the trash from each district should be shipped. Code this model in GAMS or AMPL.

District	X coord	Y coord	Trash (tons)	Fixed Cost (\$million)	Variable Cost (\$ / ton)
1	4	3	49	2	310
2	2	5	874	1	40
3	10	8	555	1	51
4	2	8	352	1	341
5	5	3	381	3	131
6	4	5	428	2	182
7	10	5	985	1	20
8	5	1	105	2	40
9	5	8	258	4	177
10	1	7	210	2	75

4. Suppose that an integer feasible solution is given for a minimum cost $n \times n$ assignment problem and that the problem is complete in the bipartite sense (so that the arc set of the problem contains n^2 arcs). Suppose that you wish to check optimality of this solution, and, in conjunction with this goal, you want a basic feasible solution (BFS) with the same flow values.

   (a) Discuss how this BFS can be constructed (that is, indicate how a set of corresponding basic arcs may be identified).

   (b) Assuming the given feasible solution is the unique optimal assignment:

      i. Is there only one BFS (set of corresponding basic arcs) associated with the feasible solution?

      ii. Show that, for nodes whose price is non-zero, there is a set of node prices of the form $\sum_{A^+(k)} c(i,j) - \sum_{A^-(k)} c(i,j)$ (assuming arc costs $c(i,j)$ and appropriately chosen arc subsets $A^+(k)$ and $A^-(k)$, where k is the node index), such that each arc will price out non-negatively for these prices.

      iii. Are the values of the node prices that prove optimality uniquely determined ? Explain.

5. Let $G = (V,E)$ be an undirected graph with costs c_{ij} for each edge $(i,j) \in E$, and let $|V| = n$. Consider the following formulation of the symmetric traveling salesman problem:

$$\min \quad \sum_{(i,j)\in E} c_{ij}x_{ij} \tag{1}$$

$$\text{subject to} \quad \sum_{j\in V:(i,j)\in E} x_{ij} = 2, \forall\, i \in V \tag{2}$$

$$\sum_{(i,j)\in E: i\in S, j\in S} x_{ij} \le |S| - 1, \forall\, S \subset V,\ |S| \ge 3 \tag{3}$$

$$x_{ij} \in \{0,1\}, \forall (i,j) \in E \tag{4}$$

Constraints (2) are the degree constraints; constraints (3) are the subtour elimination constraints.

First notice that we can add the redundant constraint

$$\sum_{(i,j)\in E} x_{ij} = n \tag{5}$$

without changing the problem.

Choose any node and designate it to be node 1. We can include subtour elimination constraints for only those subsets S that do *not* contain 1 and still ensure that the vector x defines a tour. Thus adding (5) and replacing (3) by

$$\sum_{(i,j)\in E: i\in S, j\in S} x_{ij} \leq |S| - 1, \forall\, S \subset V,\ |S| \geq 3,\ 1 \notin S, \tag{6}$$

yields a valid formulation of the TSP.

(a) Apply Lagrangian relaxation to the formulation (1), (2), (5), (6), (4) by relaxing all the degree constraints (2) *except* the degreee constraint for node 1. Write down the Lagrangian function thus obtained, using $u_i, i = 2, \ldots, n$, to denote the Lagrangian dual variables.

(b) For a given dual vector u, describe how to evaluate the Lagrangian function $L(u)$ in polynomial time.
(*Hint:* First observe that the solution to the Lagrangian subproblem must choose $n-2$ edges that have both of their endpoints in $2, \ldots, n$, and that these $n-2$ edges cannot contain a cycle.)

6. Let $(\bar{x}, \bar{u})$ be a KKT point for the problem:

$$\min \;\; f(x) \;\text{ s.t. }\; g(x) \leq b,$$

where $f : \mathbf{R}^n \longrightarrow \mathbf{R}$ and $g : \mathbf{R}^n \longrightarrow \mathbf{R}^m$, are differentiable convex functions on $\mathbf{R}^n$.

Give a lower bound in terms of $(\bar{x}, \bar{u})$ to the possibly infeasible problem:

$$\inf \;\; f(x) \;\text{ s.t. }\; g(x) \leq c,$$

where there is no relation between b and c other than both being points in $\mathbf{R}^m$.

7. This question has three parts, all of which have to be answered.

(a) You are considering operating a system in an environment of varying costs for the activities that make up the system. The state of the system is described by a vector $x \in \mathbb{R}^n$, but operational constraints require this state to satisfy conditions of the form

$$g_i(x) \leq 0, \qquad i = 1, \ldots, m, \tag{7}$$

where each g_i is a closed proper convex function on $\mathbb{R}^n$. There is a state x_0 for which $g_i(x_0) \leq 0$ for $i = 1, \ldots, m$.

The cost of operating the system for one period in a feasible state x (that is, a state x satisfying (7)) when the cost parameters are

$$c^* = (c_1, \ldots, c_n)$$

is $\gamma(c^*, x) := \langle c^*, x \rangle$. You are given that for each vector $c^* \in \mathbb{R}^n$ the cost $\gamma(c^*, x)$ is bounded above on the set of feasible state vectors x.

Is the set of feasible state vectors bounded? If so, prove it; if not, give a counterexample.

(b) The cost in (a) depends on the vector c^*. Suppose we let $v(c^*)$ be the supremum of $\gamma(c^*, x)$ over all feasible x. From the information given in (a), you know that $v(c^*)$ is finite for each $c^* \in \mathbb{R}^n$.

What can you say about continuity of the function v? Justify your answer.

(c) You have learned that it is possible to change some of the constraints on x, possibly at a cost. The result of this would be to change the right-hand sides of the inequalities (7) from 0 to b_i, where the b_i are the components of some resource vector $b \in \mathbb{R}^m$. Now the function $v(c^*)$ used in (b) depends on b too, so we can write it as $w(c^*, b)$, where the $v(c^*)$ of (b) is now $w(c^*, 0)$.

For a given c^*, what can you say about the finiteness of $w(c^*, b)$ for resource vectors $b \geq 0$? Justify your answer.

8. Consider the real-valued function defined by

$$\theta(x) := \frac{1}{2}x^2 + \sin(x)$$

for $x \in \mathbf{R}$. Prove that θ has a unique global minimizer, exhibit an interval in which this minimizer lies, and give an algorithm and (if your algorithm requires it) a starting point from which the algorithm will produce a sequence converging to the minimizer, as well as a stopping criterion that will guarantee that the the approximate solution obtained is within 10^{-5} of the actual minimizer. Prove your statements.