Theoretical Computer Science
Qualifying Exam
Fall 2007

Directions. You have four hours. Please answer questions 1 and 2 and two of the questions 3, 4, and 5. If you cannot completely answer a question, we will award partial credit for results that are true and relevant to the question, e.g. a less efficient algorithm that still works.

1. Show that the following problem is NP-hard.

   You are given a digraph G with nonnegative integer weights on the edges, two vertices s and t of G, and an integer w. You need to find out whether there exists a path from s to t of weight exactly w. The path need not be simple – it can have repeated vertices or even repeated edges.

2. For a complexity class C, define $\#C$ to be the class of functions $f : \Sigma^* \to \mathbf{N}$ for which there exists a predicate $R \in C$ and a constant c such that for every string x of length n,

   $$f(x) = |\{y \in \Sigma^{n^c} : R(x, y)\}|.$$

   Show that $\#\text{NP} \subseteq \#\text{coNP}$.

3. The following question deals with context-free grammars.
   a) Construct a family of grammars $(G_n)_n$ such that G_n is of size $O(n)$, generates exactly one string (i.e., $|L(G_n)| = 1$), and the unique string generated by G_n is of length $2^{\Omega(n)}$.
   b) Devise a randomized polynomial-time algorithm for the following problem: Given two grammars, decide whether each generates exactly one string and that that string is the same for both grammars.

4. The intersection of Monroe and Harrison streets has a crosswalk. To aid pedestrians, the city has put barrels with red flags at each side of the crosswalk. A pedestrian who crosses takes a flag and carries it to the barrel on the other side.
   a) Suppose each barrel starts with n flags. Suppose also that pedestrians arrive at random (that is, equal probability of crossing one way or the other). The time (number of crossings) at which some barrel is empty is a random variable. Compute its mean value.
   b) Many of the shops are on the south side of the street, and the merchants on this side are worried that the north barrel will run out first, thereby preventing a potential customer from crossing the street. Suppose the barrel on the north side starts with n_1 flags, and the barrel on the south side starts with n_2 flags. There is still equal probability of crossing in either direction. Compute the probability that the north barrel empties before the south one does.

5. Given a directed acyclic graph $G = (V, E)$ with weights c_e on edges, source s and sink t, an edge-saturating s-t flow is a flow from s to t such that any edge e in E carries a flow at least c_e. The minimum saturating flow problem asks for an edge-saturating flow of minimum total amount.
   a) Give a polynomial-time combinatorial algorithm for solving this problem. You may assume that edge weights c_e are integral.
   b) Prove the "min-flow max-cut" theorem, that is, the minimum saturating s-t flow is exactly equal to the maximum (directed) s-t cut in the graph.