Fall 2010 Qualifier Exam:
OPTIMIZATION

September, 2010

GENERAL INSTRUCTIONS:

1. Answer each question in a separate book.
2. Indicate on the cover of *each* book the area of the exam, your code number, and the question answered in that book. On *one* of your books list the numbers of *all* the questions answered. *Do not write your name on any answer book.*
3. Return all answer books in the folder provided. Additional answer books are available if needed.

SPECIFIC INSTRUCTIONS:

Answer 5 out of 8 questions.

POLICY ON MISPRINTS AND AMBIGUITIES:

The Exam Committee tries to proofread the exam as carefully as possible. Nevertheless, the exam sometimes contains misprints and ambiguities. If you are convinced a problem has been stated incorrectly, mention this to the proctor. If necessary, the proctor can contact a representative of the area to resolve problems during the *first hour* of the exam. In any case, you should indicate your interpretation of the problem in your written answer. Your interpretation should be such that the problem is nontrivial.

1. Suppose a manufacturer makes two products (call them "1" and "2") that both require the use of a raw material (call it "3"). Suppose that two units of raw material are needed to make each unit of product 1, and two units of raw material are also needed for each unit of product 2. The manufacturer can charge \$3 for each unit of product 1 and \$4 for each unit of product 2, and the raw material costs \$ t per unit (where t is a parameter). Because of equipment constraints, no more than 8 units of product 1 can be manufactured. In addition, there are 10 units of labor available, and each unit of product 1 requires 2 units of labor while each unit of product 2 requires 1 unit. Finally, it makes sense to consider only nonnegative quantities of both products and the raw material.

   (a) Write down a linear program, parametrized by the raw material cost t, that the manufacturer can solve to maximize his profit (defined as the difference between total revenues on the two products and total raw material costs).

   (b) Solve this linear program for all values $t \in [0, +\infty)$ and interpret the results. In particular, indicate how much profit is made for each value of t. (Hint: This is easier if you eliminate the variable that quantifies the amount of raw material purchased.)

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2. Compute the optimal solution value for each of the following linear programs. Prove or justify your answers. (In all cases the decision variable is n-dimensional).

   (a) Let $c \in \mathbb{R}^n$.

$$\begin{array}{ll} \min & c^T x \\ \text{subject to} & \sum_{i=1}^n x_i = 1 \\ & x \geq 0 \end{array}$$

   (b) Let A be a non-singular $n \times n$ matrix and $b, c \in \mathbb{R}^n$.

$$\begin{array}{ll} \min & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

   (c) Let k be an integer between 1 and n.

$$\begin{array}{ll} \min & c^T x \\ \text{subject to} & \sum_{i=1}^n x_i = k \\ & 0 \leq x \leq 1 \end{array}$$

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3. In this problem, we are scheduling a set of power plants P over a set of time periods $T = \{1, 2 \ldots, |T|\}$ to meet power demand d_t in each period. If a power plant $p \in P$ is turned on, it is able to produce any amount of power in the numerical range given by values $[\ell_p, u_p]$ $\forall p \in P$. The cost (per MWh) of producing energy at plant $p \in P$ is c_p. The plants P are of two types—nuclear plants N and coal-fired plants C. ($P = N \cup C$).

- The objective is to minimize the total cost of delivering energy over the time horizon.
- Power demand *must be* met in each period.
- To meet environmental regulations, if 2 or more coal generation plants are operating in period t, then at least 3 nuclear generating plants must also be operating in period t.
- In order to not cause a meltdown, if a nuclear generator $i \in N$ is initially turned on (started up) in period $t \in T$, then it must also be turned on for the subsequent 3 time periods.

(a) Write an *algebraic* description of the power generation model that achieves the objective and obeys all the problem restrictions. Be sure to clearly define your decision variables.

(b) If time permits, demonstrate how your model would be implemented in either the GAMS or AMPL modeling language. You may assume that all sets and parameters will be instantiated (filled in with actual values) outside of the code you write.

Hint: If $m \leq \sum_{j \in N} a_j x_j - b \leq M$, and ϵ is sufficiently small, then logical statements can be transformed into algebraic statements using the relations

$$\left(\delta = 1 \Rightarrow \sum_{j \in N} a_j x_j \leq b\right) \Leftrightarrow \left(\sum_{j \in N} a_j x_j + M\delta \leq M + b\right)$$

$$\left(\sum_{j \in N} a_j x_j \leq b \Rightarrow \delta = 1\right) \Leftrightarrow \left(\sum_{j \in N} a_j x_j - (m - \epsilon)\delta \geq b + \epsilon\right)$$

$$\left(\delta = 1 \Rightarrow \sum_{j \in N} a_j x_j \geq b\right) \Leftrightarrow \left(\sum_{j \in N} a_j x_j + m\delta \geq m + b\right)$$

$$\left(\sum_{j \in N} a_j x_j \geq b \Rightarrow \delta = 1\right) \Leftrightarrow \left(\sum_{j \in N} a_j x_j - (M + \epsilon)\delta \leq b - \epsilon\right)$$

4. Consider the following stochastic program:

$$\min_{x\geq 0} \phi(x) \stackrel{\text{def}}{=} 2x + \mathbb{E}_\omega Q(x,\omega),$$

where

$$Q(x,\omega) = \min_{y\geq 0}\{\omega y \mid y \geq 1 - x\}.$$

The random variable ω has only two outcomes, taking value 1 with probability 3/4, and value 3 with probability 1/4.

The optimal solution to this stochastic program has value $x^* = 0$

(a) What is the expected value of perfect information (EVPI)?

(b) What is the value of the stochastic solution (VSS)?

(c) What is $\partial\phi(x), \forall x \geq 0$

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5. Let $G = (V, A)$ be a directed graph, let $s, t \in V$, and let $c_{ij} \geq 0$ for arc $(i, j) \in A$ be costs on the arcs. Consider the following integer programming formulation of the problem of finding the shortest path from s to t that uses at most k arcs:

$$\min \sum_{(i,j)\in A} c_{ij}x_{ij}$$

$$\text{s.t.} \sum_{j\in\delta^+(i)} x_{ij} - \sum_{j\in\delta^-(i)} x_{ji} = \begin{cases} +1, & i = s \\ -1, & i = t \\ 0, & i \neq s,t \end{cases} \tag{1}$$

$$\sum_{(i,j)\in A} x_{ij} \leq k \tag{2}$$

$$x_{ij} \in \{0,1\}, \quad (i,j) \in A$$

where $\delta^+(i) = \{j \in V \mid (i,j) \in A\}$ and $\delta^-(i) = \{j \in V \mid (j,i) \in A\}$.

(a) Write the Lagrangian relaxation and corresponding Lagrangian dual problem obtained by dualizing the cardinality constraint (2) and comment on the difficulty of solving the Lagrangian relaxation subproblem for a fixed relaxation multiplier.

(b) Write the Lagrangian relaxation and corresponding Lagrangian dual problems obtained by dualizing the shortest path constraints (1) and comment on the difficulty of solving the Lagrangian relaxation subproblem for a fixed set of relaxation multipliers.

(c) Let $\hat{z}_1$ be the optimal value of the Lagrangian dual from part (a), $\hat{z}_2$ be the optimal value of the Lagrangian dual from part (b), and $\hat{z}_{LP}$ be the optimal value of the linear programming relaxation of the above integer programming formulation. Compare $\hat{z}_1$

to $\hat{z}_{LP}$ and justify your answer: i.e., can you say $\hat{z}_1 \leq \hat{z}_{LP}$, $\hat{z}_1 \geq \hat{z}_{LP}$, or $\hat{z}_1 = \hat{z}_{LP}$? Also compare $\hat{z}_2$ and $\hat{z}_{LP}$. (In providing your justifications, you may use the theory of Lagrangian relaxation, i.e., you do not have to provide a direct proof.)

(d) Let $LR^1(\mu)$ be the optimal value of the Lagrangian relaxation problem in part (a), where μ is the multiplier on the relaxed constraint. Is $LR^1(\mu)$ convex or concave? Prove your answer.

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6. Consider the problem

$$\begin{array}{ll} \min & -2x_1 - x_2 \\ \text{subject to} & -x_2 + 1 \geq 0 \\ & -4x_1 - 6x_2 + 7 \geq 0 \\ & -10x_1 - 12x_2 + 15 \geq 0 \\ & -x_1 - x_2^2/4 + 1 \geq 0 \\ & x_1 \geq 0, x_2 \geq 0 \end{array}$$

(a) Is this a convex optimization problem?

(b) Let X represent the linear constraints of this problem, where $X = \{x : Ax - b \geq 0\}$. Write down X, A, and b and the logarithmic barrier function $U(x,r)$ for the objective and nonlinear constraint so the the original problem is approximated by

$$\min_{x \in X} U(x,r)$$

for some scalar r.

(c) Let $x_0 = (0,1)$ and $\mathcal{A}(x_0)$ represent the constraints that are active in X at x_0. Write down $\mathcal{A}(x_0)$ and form the projection matrix P that projects any vector d onto the set

$$Z = \{z : A_{i\cdot} z = 0, i \in \mathcal{A}(x_0)\}$$

(d) Determine a feasible point for the (Wolfe) dual of

$$\min_{z \in Z} U(x_0 + z, r)$$

(e) What does this point tell you about which constraint needs to be dropped from the active set at x_0 to proceed in a gradient projection algorithm?

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7. You are investigating a function f on $\mathbb{R}^n$ about which the only thing you know is that it is proper and convex. There is a computational procedure which, if you apply it at a point $x \in \mathbb{R}^n$ at which f is finite, will give you the function value $f(x)$ and a subgradient x^* of f at x. If f is not finite at x, the procedure fails. You would like to minimize this function, but as a first step you are investigating lower bounds.

   Assume that you have used the computational procedure to produce triples

$$[x_i, x_i^*, f(x_i)], \qquad i = 1, \ldots, I.$$

   Using only these data and any computed values you derive from them, give a lower bound for f on $\mathbb{R}^n$ of the form $f(x) \geq \alpha + \beta\|x\|$. You must give explicit values for α and β in terms of the given data. Explain what properties of the data would produce a β that is (a) negative, (b) zero, or (c) positive.

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8. Consider the following nonlinear program:

$$\min_x f(x) \text{ subject to } c(x) \geq 0,$$

   where $f : \mathbf{R}^n \to \mathbf{R}$ and $c : \mathbf{R}^n \to \mathbf{R}^m$ are smooth functions, and its ℓ_1-penalized counterpart

$$\min_{(x,t)} f(x) + \mu e^T t \text{ subject to } c(x) + t \geq 0, \ t \geq 0,$$

   where $\mu \geq 0$ is a penalty parameter and $e = (1, 1, \ldots, 1)^T$.

   (a) Write down the KKT conditions for both problems.

   (b) Suppose that x^* is a KKT point for the first problem with optimal multipliers λ^*. Under what condition on μ is the same pair (x^*, λ^*) together with $t^* = 0$ a KKT point for the penalized formulation? If this condition holds, what are the optimal multipliers for the constraints $c(x) + t \geq 0$ and $t \geq 0$?

   (c) Suppose in addition to the assumptions of (b) that *strict complementarity* is satisfied for the first problem by x^* and λ^*. Under what conditions on μ will strict complementarity also be satisfied at the corresponding KKT point for the penalized formulation?

   (d) Consider now the following quadratic-penalty formulation:

$$\min_{(x,t)} f(x) + \frac{1}{2}\mu t^T t \text{ subject to } c(x) + t \geq 0, \ t \geq 0,$$

   where $\mu > 0$ is again a penalty parameter. Suppose that x^* is a KKT point for the original problem with optimal multipliers λ^*, and that $\lambda^* \neq 0$. Under what conditions on μ is the same (x^*, λ^*) together with $t^* = 0$ a KKT point for the quadratic-penalty formulation?
