Theory Qual: Fall 2013

Please answer all questions. TWO pages.

1. Let $*$ denote the Kleene star operator. If L is a language, then

$$\begin{aligned} L^* &= \{w : \exists n \geq 0, \text{ and } \exists w_1, ..., w_n \in L \text{ such that } w = w_1 \cdots w_n\} \\ &= \{\epsilon\} \cup L \cup (L \cdot L) \cup \ldots, \end{aligned}$$

where $\cdot$ denotes concatenation. We say a complexity class $\mathcal{C}$ is closed under $*$, if whenever a language L is in $\mathcal{C}$, so is L^*.

Part A. Show that the complexity class **NL** is closed under $*$.

Part B. Show that the complexity class **P** is closed under $*$.

Part C. Show that the complexity class **coNP** is closed under $*$.

2. The management of Club Zed, an exclusive singles resort, has implemented a matchmaking service for tennis players. Each player ranks his or her skill on a scale from 0 (worst) to 10 (best), and seeks an opponent with skill within δ of that value. In line with the social agenda of Club Zed, this opponent should be of the opposite gender (so men play women, or vice versa).

Part A. Design an efficient algorithm that finds a largest possible set of compatible matches (each matched player should play exactly one other player). When presented with a list of n men and n women, what is the running time of your algorithm in terms of n? You should try to find the most efficient algorithm you can.

Part B. What happens if the gender requirement is dropped?

3. Define the Clique Problem. You may use the fact that this problem is NP-complete.

Given a bipartite graph, we want to find a subgraph, with vertex sets A and B respectively from each side of the bipartite graph, such that the subgraph induced by the vertex set $A \cup B$ is a complete bipartite graph, and the size of the edge set $|A| \times |B|$ is maximized. Show that this problem is NP-hard.

4. Let

$$A = \begin{bmatrix} 1+\gamma & 1 & \ldots & 1 \\ 1 & 1+\gamma & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1\ldots & 1 & 1+\gamma \end{bmatrix}$$

be a $q \times q$ positive symmetric matrix, whose rows and columns are indexed by $\{0, \ldots, q-1\}$, and where the diagonal entries are $1+\gamma$ and the off-diagonal entries are 1. The function P_A is defined for any graph $G = (V, E)$, as follows

$$P_A(G) = \sum_{\sigma: V \to \{0,\ldots,q-1\}} \prod_{e=\{u,v\} \in E} A_{\sigma(u),\sigma(v)}.$$

Part A. For any subset $S \subseteq E$, let $\kappa(V, S)$ be the number of connected components of the graph with vertex set V and edge set S.

Prove that

$$P_A(G) = \sum_{S \subseteq E} q^{\kappa(V,S)} \gamma^{|S|}.$$

Part B. Let $q = 2$. Suppose you are given a blackbox such that for any graph G, it allows you to sample 0-1 assignments σ proportional to the relative weight $\prod_{\{u,v\} \in E} A_{\sigma(u),\sigma(v)}$. Give a (randomized) polynomial time approximation algorithm to compute $P_A(G)$, for any graph G, using the sampling blackbox. For input G and $\epsilon > 0$, your algorithm should produce an approximate value that is within $(1 \pm \epsilon)$ of $P_A(G)$, with probability at least 3/4. Your algorithm should run in time polynomial in the size of G and $1/\epsilon$. Note that the sampling blackbox can be applied to graphs other than the input G. Your algorithm is measured in terms of the number of ordinary operations plus the number of calls to the sampling blackbox.

Good Luck!