MATHEMATICAL PROGRAMMING

Instructions for Depth Exam Students: Answer 6 of the following questions, at most 2 from the Linear Programming Section.

Instructions for Breadth Exam Students: Answer 3 of the following questions, at most 2 from the Linear Programming Section.

Linear Programming Section (Answer at most 2 questions from this section)

1. Solve the following problem:

$$\begin{array}{lccccccccc}
\text{maximize} & & & & 2y_1 - y_3 & & & & & \\
 & 5y_1 & - & 2y_2 & + & y_3 & - & y_4 & = & 36 \\
\text{subject to} & y_1 & & & + & y_3 & & & \geq & 4 \\
 & y_1 & + & 3y_2 & + & y_3 & & & \geq & 1 \\
 & y_1 \leq 8, & & y_2 \geq 0, & & y_3 \leq 0, & & y_4 \geq 0 & &
\end{array}$$

   Does the problem have a unique optimal solution? Justify.

2. The optimal solution of the LP problem

$$\begin{array}{ll}
\text{maximize} & z = 3x_1 - 12x_2 + 2x_3 \\
 & x_1 - 3x_2 + x_3 \leq 2 \\
\text{subject to} & x_1 - x_2 - x_3 \leq 3 \\
 & x_i \geq 0, i = 1, 2, 3
\end{array}$$

   is $x_1^* = 2, x_2^* = x_3^* = 0$.

   (a) Find the ranges over which you can vary the cost coefficients of x_1 and x_2 individually without changing the solution.

(b) Using the given optimal solution, find the optimal solution of the original problem with the new constraint

$$2x_1 + 5x_2 + x_3 \leq 3.$$

3. Consider the following LP's:

$$(LP_1) \qquad \min_{x} \quad cx$$
$$\text{s.t.} \quad Ax = b, \;\; x \geq 0$$

$$(LP_2) \qquad \min_{x,\, y} \quad cx + ey$$
$$\text{s.t.} \quad Ax + y = b$$
$$x, y \geq 0,$$

where $x \in \mathbb{R}^{n}, y \in \mathbb{R}^{m}$, and $e = (1, \ldots, 1)$.

(a) Suppose that the simplex method is used on LP_2 and terminates with satisfaction of the unboundedness criterion. What conditions on the final tableau (or dictionary) or analytic conditions are sufficient to guarantee the unboundedness of LP_1?

(b) If a final tableau (or dictionary) establishes unboundedness of LP_2 but does not satisfy the conditions of part (a), describe an efficient approach (starting with the final basis generated for LP_2 and solving at most a single LP) for determining the feasibility of LP_1.

In the case that no additional LP solution is required to establish the feasibility of LP_1, what can be said about the existence of an optimal solution of LP_1?

Advanced Topics Section

4. Consider the convex program

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g(x) \leq 0 \end{array}$$

where $f\colon \mathbb{R}^n \to \mathbb{R}$ and $g\colon \mathbb{R}^n \to \mathbb{R}^m$ are differentiable, convex functions on $\mathbb{R}^n$. Let the feasible region $X\colon= \{x \mid g(x) \leq 0\}$ be nonempty and let

$$\nabla g_I(\bar{x})z \leq 0, \nabla f(\bar{x})z < 0 \text{ have } \textbf{no} \text{ solution } z \in \mathbb{R}^n$$

for some nonempty $I \subset \{1, \ldots, m\}$ and some $\bar{x} \in \mathbb{R}^n$ not necessarily in X. Construct a linear program that provides a lower bound to $\inf \{f(x) \mid g(x) \leq 0\}$. Prove that the linear program is solvable and give an explicit expression for your lower bound in terms of $\bar{x}$ and a solution to your linear program.

5. (a) Construct an augmented Lagrangian for the nonlinear program

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g(x) \leq 0 \end{array}$$

where $f\colon \mathbb{R}^n \to \mathbb{R}$ and $g\colon \mathbb{R}^n \to \mathbb{R}^m$ are differentiable functions on $\mathbb{R}^n$.

(b) Relate an **unconstrained** stationary point of the augmented Lagrangian to the original nonlinear program. Establish your claim.

(c) Give one step of the augmented Lagrangian algorithm, or any other algorithm that utilizes the augmented Lagrangian formulation.

6. Consider the quadratic program

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}x^T Q x + c^T x \\ \text{subject to} & Ax = b \end{array}$$

Prove that x^* is a local solution if and only if x^* is a global solution. [Hint: Consider first and second order necessary conditions]

7. Let f be a convex function and let x be a point of $\operatorname{ri}\operatorname{dom} f$. Assume that $f(x) > -\infty$, and write L for the subspace parallel to $\operatorname{dom} f$.

   (a) Show that $\partial f(x) \neq \emptyset$.

   (b) Let $D = \partial f(x) \cap L$. Show from first principles that D is nonempty, compact, and convex. (You may assume elementary results about dimensionality, relative interiors, orthogonal decompositions, etc.)

   (c) Show that $\partial f(x) = D + L^{\perp}$.

8. Let G be a directed graph with specified edge lengths $d_{ij} > 0$ for the m edges (i, j). You may assume that d_{ij} is defined for all $i, j = 1, \cdots, n$ with $i \neq j$.

   (a) Formulate the problem of determining the shortest directed path from node 1 to node n as a network flow problems of the form:

$$\begin{array}{ll} \min & cx \\ \text{s.t.} & Ax = b \\ & x \geq 0, \end{array}$$

where A is a node-arc incidence matrix. (Be sure to explain the correspondence between an appropriate optimal solution x^* and a directed path.)

(b) What is the relationship between an appropriate set of optimal dual variables for the network flow problem in part (a) and path lengths for the original digraph? (Be precise.)

9. Consider the general fixed charge problem

$$\begin{array}{ll} \text{minimize} & \langle c, y\rangle + \langle d, x\rangle \\ \text{subject to} & \begin{array}{l} y \in Y := \{y : Ay = b,\ y \geq 0\} \\ x_j = 0 \text{ if } y_j = 0,\ x_j = 1 \text{ if } y_j > 0 \quad \forall j = 1, \ldots, n \end{array} \end{array}$$

where A is a mxn real matrix and $\text{rank}(A) = m$. Show that if every vertex of Y is nondegenerate and all fixed costs are equal (i.e. $d_j = r$ for all $j = 1, \ldots, n$), then any optimal vertex solution of the linear program

$$\begin{array}{ll} \text{minimize} & \langle c, y\rangle \\ \text{subject to} & y \in Y \end{array}$$

solves the fixed charge problem, by providing an optimal value for y.