Numerical Analysis   Depth Examination   Fall 1995

You should answer five out of the eight problems below.

·• 1 •· This problem has two separate parts dealing with numerical integration.

(a) You are asked to use numerical integration in order to approximate the integral

$$\int_{-1}^{1} \frac{|x^2 - 1/4|^3 - 1/2}{\sqrt{1-x^2}}\, dx.$$

Design a scheme that, in your opinion, can be used for an efficient estimation of the integral. Justify your choice, based on the known error analyses of numerical integration schemes. *Remark.* Your answer should be qualitative, that is, if you suggest a certain composite rule, you need not discuss the number of subintervals to be used.

(b) Show that the composite Simpson's rule is obtained from the composite trapezoid rule by extrapolation.

·• 2 •·

(a) You are given a real $n \times n$ matrix A, and you are told that the dominant eigenvalue λ_0 of A is not real. Suggest an algorithm that finds an eigenpair (λ_0, v) of A.

(b) State a condition that guarantees the success of your algorithm.

(c) Consider the matrix

$$B = \begin{pmatrix} -5 & 1 & 0 & 0 \\ a & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

with $1 \le a \le 3$. Is it possible to tell whether the dominant eigenvalue of B is real? Is it possible to tell whether it is not real?

·• 3 •· Consider the initial-value problem

$$\begin{aligned} y' &= 3(1-x)y, \qquad 0 \le x \le 4, \\ y(0) &= 1 \end{aligned}$$

and the numerical method

$$Y_{k+1} = Y_{k-1} + 2h[3(1-x_k)Y_k]$$

with $Y_0 = 1$, $Y_1 = 1 + 3h$, and $x_k = kh$.

(a) Discuss the convergence or non-convergence of this method.

(b) Discuss the behavior of the error.

·• 4 •· Consider the boundary-value problem

$$\begin{aligned} u^{iv} + uu''' &= x \\ u(-1) = u'(-1) &= 0 \\ u(1) = u'(1) &= 0 \end{aligned}$$

Discuss the numerical solution of this problem using the 'shooting' method.

·• 5 •· Consider the partial differential equation $u_t + au_x = 0$. Show that the Lax-Wendroff scheme has the highest accuracy among all stable finite difference schemes of the form

$$v_m^{n+1} = Av_{m-1}^n + Bv_m^n + Cv_{m+1}^n$$

which approximate this partial differential equation.

·• 6 •· Consider the finite difference scheme

$$v_m^{n+1} = \frac{1}{2}\left(v_{m+1}^n + v_{m-1}^n\right) - \frac{\lambda}{2(1+\lambda^2)}\left(v_{m+1}^n - v_{m-1}^n\right)$$

for the equation $u_t + u_x = 0$ where $\lambda = \Delta t/\Delta x$ and v_m^n is an approximation to $u(t_n, x_m)$.

(a) Show that this scheme is stable for all values of λ.

(b) For the initial value problem, discuss the convergence of the solution of the scheme to the solution of the differential equation.

·• 7 •·

(a) State Taylor's formula with integral remainder (for the Taylor expansion, truncated after terms of degree $< k$, of some function f at some point a) and interpret it as a pointwise error formula for the Hermite interpolant from $\Pi_{<k}$ to f at a.

(b) Where in the error formula in (a) does there appear a right inverse for the linear map $f \mapsto D^k f$ of k-fold differentiation, and what is it?

(c) Give an abstract version of the above formula, involving some bounded linear projector P (instead of Hermite interpolation) on some normed linear space X, and some bounded linear map $E : X \to Y$ with $\operatorname{ran} P = \ker E$ (instead of D^k).

·• 8 •·

(a) Define what it means for a linear map, from some normed linear space to some normed linear space, to be *bounded below*, and relate it to the property of the map having a bounded inverse.

(b) Give an example of a bounded linear map which is bounded below yet not invertible.

(c) Give an example of a bounded linear map which is invertible yet not bounded below.

(d) State the basic theorem in Functional Analysis which implies that a bounded linear map, from some Banach space to some Banach space, is boundedly invertible if and only if it is 1-1 and onto.