Fall 1999 Qualifier Exam:
MATHEMATICAL PROGRAMMING

Instructions: Answer 5 of the following 7 questions.

1.

$$\min \ \frac{3x_1^2}{2} - x_1 x_2 + x_2^2 + 6x_1 - x_2$$

subject to:

$$\begin{array}{ccccc} x_1 & + & x_2 & \geq & 2 \\ x_1 & , & x_2 & \geq & 0 \end{array}$$

What is the minimum value and where is it attained?

2. A number of power stations are committed to meeting the following electricity load demands over a day:

12 pm to 6 am	15,000 megawatts
6 am to 9 am	30,000 megawatts
9 am to 3 pm	25,000 megawatts
3 pm to 6 pm	40,000 megawatts
6 pm to 12 pm	27,000 megawatts

There are three types of generating units available, twelve of type 1, ten of type 2, and five of type 3. Each generator has to work between a minimum and a maximum level. There is an hourly cost for each generator at minimum level. In addition there is an extra hourly cost for each megawatt at which a unit is operated above minimum level. To start up, a generator also involves a cost. All this information is given in the table below:

	Min Lev	Max Lev	Cost at minimum	Cost per MW above minimum	Startup cost
Type 1	850 MW	2000 MW	1000	2	2000
Type 2	1250 MW	1750 MW	2600	1.3	1000
Type 3	1500 MW	4000 MW	3000	3	500

In addition to meeting the estimated load demands there must be sufficient generators working at any time to make it possible to meet an increase in load of up to 15%. This increase would have to be accomplished by adjusting the output of generators within their permitted limits.

Construct a (GAMS or AMPL) model whose solution would tell you which generators would be working in which periods of the day to minimize total cost. Be careful to be precise in your model formulation, particularly with the equations and variables that define the model.

3. Suppose that an $N \times N$ grid is given, and is to be partitioned into N subdomains with the properties that each subdomain is of equal size and the total number of internal edges of the partition is as large as possible. (An internal edge is defined as one that is not part of the boundary between two subdomains or the boundary of the original grid.)

   (a) Model this problem as a linear integer programming problem using the binary variable x_i^p to represent the decision of whether or not to assign grid cell i to subdomain p . Discuss why your model correctly represents the given problem.

   (b) Consider the continuous relaxation of the problem in part (a) . Discuss whether or not the solution corresponding to setting all assignment variables to $1/N$ is feasible for the relaxation. If this solution is feasible for the relaxation, determine its objective value.

   (c) Discuss the relationship between your best estimates for the optimal values of the original integer program and its continuous relaxation.

4. Suppose that a network flow (minimization) problem is given with a linear objective with non-negative coefficients, non-negativity constraints on the variables, but with divergence equations specified at only a proper subset N' of the node set N (and no other constraints). (A divergence equation at a node i is a constraint of the form: total flow out of node i - total flow into node i = constant).

   (a) If this problem has an optimal solution, what is an optimal flow value for any arc internal to the complement of N' (that is, an arc incident only to such nodes)?

(b) Discuss how this problem may be converted to a equivalent standard network flow problem with divergence equations at all nodes (be sure to discuss how the nodes and arcs in the new problem are related to the nodes and arcs of the original problem – do not assume that these sets coincide). What is the rank of the matrix in the set of equality constraints in the new problem if the graph of the original problem was connected (in the usual undirected path sense) ?

5. Consider the nonlinear program:

$$\min_{x \in \mathbf{R^n}} \{f(x) \mid g(x) \leq 0\},$$

where $f : \mathbf{R^n} \longrightarrow \mathbf{R^1}$ and $g : \mathbf{R^n} \longrightarrow \mathbf{R^m}$, are convex functions on $\mathbf{R^n}$. Suppose that f is nonnegative on $\mathbf{R^n}$ and that $g(\hat{x}) < 0$ for some $\hat{x}$ in $\mathbf{R^n}$. Give an upper bound in terms of $\hat{x}$ on the 1-norm, $\|\bar{u}\|_1$, of any optimal Lagrange multiplier $\bar{u} \in \mathbf{R^m}$ associated with any solution $\bar{x}$ to the problem.

6. Consider the problem of minimizing (locally) the function

$$f(x, y) = (1/2)\{p(x^2 + y^2 - 2x - 2y) + (xy - 1)^2\},$$

where x and y are real numbers and p is a real parameter. Answer the following questions, justifying your answers. You may use known theorems, but if you do then you must clearly describe them.

(a) What are values x_0 and y_0 such that f has a stationary point at (x_0, y_0) for *every* value of p?

(b) For which value(s) of p does (x_0, y_0) satisfy the second-order necessary condition?

(c) For which value(s) of p does (x_0, y_0) satisfy the second-order sufficient condition?

(d) For which values(s) of p can you be certain that f is convex in a neighborhood of (x_0, y_0)?

(e) For which value(s) of p can you be certain that Newton's method, if started sufficiently close to (x_0, y_0), will converge quadratically to (x_0, y_0)?

7. Suppose f is a closed convex function on $\mathbf{R^2}$ that has finite values at the origin and at the points $(1,1)$, $(1,-1)$, $(-1,-1)$, and $(-1,1)$. Complete the following requirements, justifying your answers. If you appeal to general theorems, describe them carefully.

   (a) Show that f is proper.

   (b) Let f^* denote the conjugate function of f. Show that whenever p is a vector in $\mathbf{R^2}$ with $\|p\| < 1$, there exists an $x^*_p \in \mathbf{R^2}$ that minimizes the function $f^*(x^*) - \langle x^*, p\rangle$.

   (c) Let g be the recession function of f^*. Show that for each $x^* \in \mathbf{R^2}$, $g(x^*) \geq \|x^*\|$, where the norm is the Euclidean norm.