MATHEMATICAL PROGRAMMING

Spring 2000 Qualifying Exam
February 8, 2000
Instructions: Answer 5 of the following 8 questions.

1. Solve the following quadratic program:

$$\begin{array}{ll} \max & -1/2x_1^2 + 6x_1 + x_2 \\ \text{subject to} & x_1 + x_2 \le 5 \\ & 2x_1 + x_2 \le 6 \\ & x_1, x_2 \ge 0 \end{array} \tag{1}$$

   Use your solution to answer the following questions:

   (a) Is the solution of (1) unique? Justify.

   (b) What is the optimal solution if we add the single constraint $3x_1 + x_2 \le 10$ to (1)?

   (c) What is the optimal solution if we add the single constraint $x_1 - x_2 \ge 6$ to (1)?

2. Consider the quadratic least 2-norm formulation of a linear program:

$$\min_x c'x + \frac{\varepsilon}{2}x'x \;\; \text{s.t.} \;\; Ax \ge b,$$

   where A is an $m \times n$ matrix, ε is a sufficiently small positive number and $'$ denotes the transpose.

   (a) By using duality theory, reduce the problem to that of minimizing a positive semidefinite quadratic function in m nonnegative variables and no other constraints.

   (b) Suppose $AA' = I$, where I is the identity matrix. What is the solution x?

3. Consider the nonlinear program:

$$\min_x f(x) \;\; \text{s.t.} \;\; g(x) \le 0,$$

and the associated exact penalty minimization problem:

$$\min_{x \in R^n} f(x) + \alpha \|(g(x))_+\|_1.$$

Here, $f : R^n \longrightarrow R$, $g : R^n \longrightarrow R^m$, α is a nonnegative real number, $\|\cdot\|_1$ is the 1-norm (sum of absolute values) and $(\cdot)_+$ when applied to a vector of real numbers, replaces negative components by zeros. Enumerate precisely the simplest conditions on f, g and α that are needed to guarantee that a solution $\bar{x}$ of the nonlinear program also solves the exact penalty problem. Prove your claim.

4. A user named X (who has not had CS 730) is trying to find a global minimizer of the function defined on $\mathbf{R^3}$ by

$$f(x) = 6x_1^2 + 4x_1x_2 + 8x_1x_3 + x_2^2 + 4x_2x_3 + 4x_3^2 + 4x_1 + 2x_2 + 4x_3 + 2.$$

X decides to proceed by taking the derivative $J(x)$ of f, setting $J(x)$ equal to zero, and solving the resulting set of equations. However, use of a solution package on the equations produces an error message. At this point X begins to wonder whether this function has any minimizer, and even if it does whether there might be problems with local minimizers that are not global.

Answer the following questions about X's troubles, justifying each of your answers:

(a) Does f have a global minimizer? If so, is it unique? Are there local minimizers that are not global minimizers?

(b) What property of the equations $J(x) = 0$ might lead to an error message from a solution package?

(c) What do you think would have happened if the coefficient of x_2^2 in $f(x)$ had been 2 instead of 1?

In your answer, you may use the fact that

$$\begin{pmatrix} 6 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 1 & 2 \end{pmatrix}$$

if you find it to be of help.

5. Let f be a closed proper convex function on $\mathbf{R^n}$, and assume that $x_0 \in \mathbf{R^n}$ is a minimizer of f. Suppose you know that there is some number μ such that for each x^* in some neighborhood U of the origin in $\mathbf{R^n}$, *if* the function $f(x) - \langle x^*, x\rangle$ has a minimizer z then $\|z\| \leq \mu$.

   Show that there is some $\epsilon > 0$ such that for each x^* in the ball of radius ϵ around the origin, the function $f(x) - \langle x^*, x\rangle$ actually does have a minimizer.

6. Consider a feature selection problem in which x represents an n-vector of unknown feature weights to be used in a model of the form Ax (where A is $m \times n$) to approximate a data vector b. Let E(x) denote the error $\|Ax - b\|_1$ and $k(x)$ denote the number of non-zero elements of x. We also assume that a suitably large numerical vector U is given such that weight values satisfying $-U \leq x \leq U$ are expected to provide the best fits to the data regardless of the number of features used.

   (a) Model as a linear mixed integer programming problem the problem

   $$\min \lambda \cdot k(x) + (1 - \lambda) \cdot E(x) \qquad \text{s.t. } -U \leq x \leq U,$$

   where λ is a parameter chosen from the interval [0,1].

   (b) As λ varies from 0 to 1, what values would you typically expect to see for $k(x^*(\lambda))$, where $x^*(\lambda)$ solves the problem in (a)? In particular, discuss the form of the optimal solutions near $\lambda = 0$ and 1, and how this leads to the expected values of $k(x^*(\lambda))$. (Your answer may involve rank(A).)

7. Consider the following bipartite network flow problem:

   There are two supply nodes, with node 1 constrained to send no more than 120 units and with node 2 constrained to send no more than 90 units. The supply nodes send directly to two demand nodes 3 and 4, where node 3 requires exactly 130 units and node 4 requires exactly 70 units. In order to balance flow as much as possible on the four arcs, a target flow of 50 is imposed on each arc, so that the cost function on each arc is of the form $|f - 50|$, where f is the flow on the arc.

   (a) Model (as a standard network flow problem with linear arc costs) the problem of minimizing total deviation from the target flows subject to the supply and demand constraints.

(b) Solve via network flow techniques the problem in (a) and provide the optimal flows and optimal node multipliers (dual variables). (You should use an advanced starting solution to speed up the solution process, which otherwise will be time-consuming.)

8. A power company faces demands during both peak and off-peak times. If a price of p_1 dollars per kilowatt-hour is charged during the peak time, customers will demand $60 - 0.5p_1$ kwh of power. If a price of p_2 dollars is charged during the off peak time, customers will demand $40 - p_2$ kwh. The power company must have sufficient capacity to meet demand during both peak and off-peak times. It costs \$10 per day to maintain each kilowatt-hour of capacity. Write down an AMPL or GAMS model that determines how the power company can maximize its daily profit.

   Describe how you would update the model above if the power company is allowed to buy power (kwh) from the grid during peak hours for \$8 and during off-peak for \$6.