OPTIMIZATION

Spring 2003 Qualifying Exam
January 31, 2003
Instructions: Answer 5 out of 8 questions

1. Suppose that f is an extended-real-valued, closed proper convex function on $\mathbb{R}^n$, and that x_0 is a point in the relative interior of $\operatorname{dom} f$. We will call a direction $v \neq 0$ a *descent direction* for f at x_0 if there is some $\mu < 0$ such that for all small positive t we have $f(x_0 + tv) \leq f(x_0) + \mu t \|v\|$.

   Assume that someone has proposed the following idea for finding a descent direction: "If x^* is any nonzero element of the subdifferential $\partial f(x_0)$, the direction $-x^*$ is a descent direction for f at x_0."

   (a) Show by counterexample that this proposal is wrong (that is, produce a function f and a point x_0 for which it doesn't work).

   (b) Prove that if x_0 is not a minimizer of f and if, instead of any element of $\partial f(x_0)$, one chooses the element closest to the origin (in the Euclidean norm), then the negative of this element is a descent direction for f at x_0.

2. Let $\hat{x}$ be an arbitrary point in R^n and consider the minimization problem:

$$\min \ \ f(x) \ \text{ s.t. } \ b \leq x \leq c,$$

   where $f : R^n \longrightarrow R$ is a differentiable convex function on R^n.

   (a) Give a lower bound to the minimum value of the problem in terms of $\hat{x}$, b, c.

   (b) If $\hat{x}$ is feasible for the above problem, give your best estimate of the minimum value of the problem in terms of $\hat{x}$, b, c.

3. Consider a maximum flow problem defined on a digraph (N,A) with flow bounds $0 \leq x \leq c$, where the capacity on the feedback arc (t,s) is infinite. Assume that an optimal basic feasible solution $\hat{x}$ with positive optimal value is given.

(a) Assume that the price (dual variable) p_s for node s is 0. Show that all node prices are 0 or 1 and, using these prices, show that the forward arcs of a corresponding minimum cut match the set of arcs with reduced cost 1.

(b) Give a numerical example in which the set comprised of the arcs of $\hat{x}$ (the optimal BFS described above) that are at capacity do not form a minimum cut.

4. (a) Consider the continuous knapsack problem defined by the linear program

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^{n} a_j x_j \leq b \\ & x_j \geq 0, \end{aligned} \tag{1}$$

where b and a_j for $j = 1, \ldots, n$ are all positive integers. Give an $O(n\log(n))$ algorithm to solve this linear program.

(b) Now consider problem (1), but with the added restriction

$$x_j \in \{0,1\} \text{ for } j = 1, \ldots, n.$$

Why cannot your algorithm for (1) be modified to work for this case?

(c) Show how to reformulate the 0-1 integer program in (b) as a linear program that has $O(nb)$ constraints and $O(nb)$ variables, and for which the extreme points of the feasible region have values that are integral. (Hint: This linear program should model a longest-path problem on an acyclic graph.)

5. A cooking oil manufacturer blends a product from five raw oils, whose cost (in dollars per ton) and hardness (a physical property, measured in unspecified units) is given in the following table:

raw oil	cost ($/ton)	hardness
1	110	8.8
2	120	6.1
3	130	2.0
4	110	4.2
5	115	5.0

The final blended oil sells for \$150/ton. Equipment constraints require that the total amount of oils 1 and 2 (combined) in the final blend cannot exceed 200 tons, while the total of oils 3, 4, and 5 (combined) cannot exceed 250 tons. The final blended oil must have a hardness of between 3 and 6 units. (Hardness blends linearly.)

(a) Formulate a linear program to determine how much of each oil should be included in the blend to maximize profit subject to the given constraints.

(b) Suppose that at most 3 oils can be included in the final blend. Suppose also that if an oil is used in the blend, at least 20 tons of it must be used. By modifying the linear program above, formulate a mixed integer program that includes these restrictions.

(c) Suppose now that an additional fixed cost of \$500 is incurred for each raw oil used in the blend. However, since oils 2 and 4 come from the same supplier, a total fixed cost of \$750 (not the expected \$1000) is charged if *both* of these oils are used. Describe what additional variables and constraints, and changes to the objective, are needed to account for these additional conditions.

6. Consider the following problem:

$$\min_{x \in \mathbf{R}^n} c^T x \text{ subject to } a^T x = b, \; l \le x \le u,$$

where c, a, l, and u are all vectors in $\mathbf{R}^n$, all the components of a and c are positive, all the components of l and u are finite, and b is a positive real number.

(a) If $\tilde{x}$ is an optimal solution for this problem, prove that there exists $\tilde{\pi}$ such that $\tilde{\pi} a_j < c_j$ implies $\tilde{x}_j = l_j$, $\tilde{\pi} a_j > c_j$ implies $\tilde{x}_j = u_j$, and $l_j < \tilde{x}_j < u_j$ implies $\tilde{\pi} a_j = c_j$.

(b) Suppose the variables are listed in the problem so that

$$(c_1/a_1) < (c_2/a_2) < \ldots < (c_n/a_n)$$

Prove that if a feasible solution exists, then there is an optimal solution $\bar{x}$ satisfying the following property that for some s:

$$\begin{aligned} \bar{x}_j &= u_j, \qquad j = 1, 2, \ldots, s-1, \\ l_s \le \bar{x}_s &\le u_s \\ \bar{x}_j &= l_j, \qquad j = s+1, s+2, \ldots, n. \end{aligned}$$

7. (a) Without using the simplex method, show that $x^* = (2,0,1)$ is an optimal solution of:

$$\begin{array}{ll} \min & 4x_1 + 2x_2 - x_3 \\ \text{subject to} & 2x_1 + x_2 - x_3 \geq 3 \\ & 6x_1 + 7x_2 - 5x_3 \geq 6 \\ & 6x_1 + x_2 - 3x_3 \geq 5 \\ & -x_1 - x_2 + x_3 \geq -1 \\ & x_1, x_2, x_3 \geq 0 \end{array} \tag{2}$$

   (b) Is the solution of (2) unique? Justify.

   (c) Write down the dual problem and a solution.

   (d) Let p_2 refer to the objective coefficient multiplying x_2. What is the smallest value p_2 can take on such that x^* remains optimal?

   (e) If we replace the objective of (2) by

$$ax_1^2 + 4x_1 + 2x_2 - x_3$$

   for what range of values of a does x^* remains optimal?

   Be sure to quote any theorems you use accurately.

8. Consider the quadratic program

$$\min_x \frac{1}{2} x^T G x + d^T x \text{ s.t. } Ax = b$$

   where A has full row rank. Let Z be a basis for the null space of A.

   (a) Show that the QP has a unique global minimizer at a point x^* satisfying $Gx^* + d = A^T \lambda^*$ (for some λ^*) if and only if $Z^T G Z$ is positive definite.

   (b) What can you say about the (solutions of) QP if $Gx^* + d = A^T \lambda^*$ has a solution and $Z^T G Z$ is singular and positive semidefinite.

   (c) What can you say if either $Z^T G Z$ is indefinite, or $Gx^* + d = A^T \lambda^*$ has no solution.

   (d) Why in practice would you wish to orthonormalize Z?

   Be sure to prove your assertions and quote any results that you invoke accurately.