OPTIMIZATION

Spring 2004 Qualifying Exam
February 2, 2004
Instructions: Answer any 5 of the following 8 questions.

1. (i) Solve the linear program:

$$\min\ 6x_1 + 7x_2 - 3x_3$$

subject to:

$$\begin{array}{rrrl} -3x_1 & +2x_2 & +x_3 & \geq -6 \\ -4x_1 & -3x_2 & +x_3 & = -4 \\ -15x_1 & -5x_2 & +2x_3 & \geq -9 \\ x_1, & x_2 & & \geq 0 \\ & & x_3: & \text{free.} \end{array}$$

(ii) Write the dual to the above linear program.

(iii) By using *only* information from your answer to part (i) above, obtain a solution to the dual problem.

2. Consider a one-period model of the evolution of a financial market having finitely many asset types indexed by $a = 1, \ldots, A$. For each asset index a, the price now (at time $t = 0$) of the ath asset is a given real number q_a.

One period in the future (at time $t = 1$), the market may be in any one of a finite number of states indexed by $s = 1, \ldots, S$. For each a and s, the price of the ath asset in state s will be a given real number d_{as}.

A *portfolio* of assets (either at time 0 or at time 1) is described by a vector $w \in \mathbb{R}^A$ such that w_a gives the number of units of asset a held in the portfolio. This number may be positive (long holding), zero, or negative (short holding). Thus, if $q \in \mathbb{R}^A$ is the vector whose components are the q_a, then the price at time 0 of the portfolio described by w is the inner product $\langle w, q \rangle$.

We say that a portfolio w is a *weak arbitrage* if its price at time 0 is strictly negative and, for each state s at time 1, the value $\sum_{a=1}^{A} w_a d_{as}$ is nonnegative. (Thus, buying a weak arbitrage gives you a sure payoff now and also guarantees that you cannot lose money at time 1.)

Show that the following are equivalent:

- There is no weak arbitrage.
- There exists a system of *state prices* given by nonnegative real numbers v_s having the property that for each asset a, the current price q_a of this asset equals $\sum_{s=1}^{S} d_{as} v_s$. (Thus, the state prices determine the current prices of *all* assets as nonnegative linear combinations of their uncertain future values.)

3. The variable y represents the yield in a chemical process. There are n process variables $x_1, x_2, \ldots, x_n$ (such as temperature, flow rate, etc) which influence the yield. Data was collected to observe the yield y for various values of the process vector $x^t - (x_1^t, x_2^t, \ldots, x_n^t)$. It is believed that y can be reasonably approximated by a convex quadratic objective function. Formulate the problem of finding the best convex quadratic approximation $Q(x)$ for y using the available data as a nonlinear program in AMPL or GAMS, and discuss the important features of your formulation.

   To test your model, ensure that you write AMPL or GAMS statements to generate random inputs that are consistent with the above hypothesis. Also, write statements to print out the results of the model to show how the solution relates to your random inputs.

4. Consider the following network flow problem in which two supply centers S1 and S2 are to meet at minimum incremental cost an increased set of demands at two demand centers D1 and D2 . Current demand is satisfied with S1 supplying 5 units to D1 and S2 supplying 5 units to D2. The projected demands to be met are 7 units (an increase of 2) at D1 and 8 units (an increase of 3) at D2. S1 has 20 units of supply available, and can supply both D1 and D2 with additional units at rate $ 2 or can reduce supply to D1 with a saving of $ 1 per unit. S2 has 15 units of supply available, and can supply both D1 and D2 with additional units at rate $ 4 or can reduce supply to D2 with a saving of $ 3 per unit.

   (a) Formulate this problem as a linear network flow problem with a divergence equation at each node. (Note: a dummy node is required.)

   (b) Solve the problem, giving the optimal price at each node, and verify optimality via these prices. State the optimal solution in terms of units of demand supplied by each supply center to each demand center and state the optimal incremental cost.

5. Consider the *capacitated facility location problem* (CFL) with m customers and n facilities:

$$\begin{aligned} \min \quad & \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} + \sum_{j=1}^{n} f_j y_j && (1) \\ \text{subject to} \quad & \sum_{j=1}^{n} x_{ij} = 1, \text{ for } i = 1,\ldots,m && (2) \\ & \sum_{i=1}^{m} d_{ij}x_{ij} \le K_j y_j, \text{ for } j = 1,\ldots,n && (3) \\ & x_{ij} \in \{0,1\}, \text{ for } i = 1,\ldots,m, \ j = 1,\ldots,n; && (4) \\ & y_j \in \{0,1\}, \text{ for } j = 1,\ldots,n && (5) \end{aligned}$$

Here K_j is a positive integer representing the capacity of each potential facility j, and d_{ij} is a positive integer representing the capacity required to satisfy the demand of customer i from facility j if facility j is opened. For the questions below, consider an instance of CFL defined in part by $m = 4$, $n = 4$, and

$$d_{11} = 4, d_{21} = 6, d_{31} = 7, d_{41} = 3$$
$$K_1 = 12.$$

(You do not need to know the rest of the data to answer the questions.)

(a) Let $(\bar{x}, \bar{y})$ be a fractional solution of the LP relaxation of this instance of CFL defined in part by

$$\bar{x}_{11} = 1, \bar{x}_{21} = 1, \bar{x}_{31} = 0, \bar{x}_{41} = 0$$
$$\bar{y}_1 = \tfrac{5}{6}.$$

Write down a valid inequality for this instance of CFL that cuts off this point.

(b) Generalize the inequality you wrote down in (a) to define a family of valid inequalities for the general model (2)–(5). (State the family in general algebraic form, so that it is a family of valid inequalities for the model CFL, and *not* just for this instance.)

(c) Consider the inequality

$$x_{11} + x_{21} + x_{41} \leq 2$$

Is this inequality valid for this instance of CFL? Why or why not?

(d) Does the inequality in (c) define a facet of the convex hull of feasible solutions of this instance of CFL? Why or why not?

6. Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable functions on $\mathbb{R}^n$, and consider the following nonlinear program:

$$\min f(x) \text{ s.t. } g(x) \leq 0.$$

Suppose that $(\bar{x}, \bar{u})$ is the solution to the KKT Saddlepoint Problem for this nonlinear program, and suppose further that $f(\bar{x}) > f(\hat{x})$ for some $\hat{x} \in \mathbb{R}^n$. Give a nonzero *lower* bound on $\|\bar{u}\|_1$ in terms of $\bar{x}$ and $\hat{x}$.

7. State a result linking the penalty function

$$\phi(x, \sigma) = f(x) + \frac{1}{2}\sigma \sum_i [\min(c_i(x), 0)]^2$$

to the problem

$$\min f(x) \text{ subject to } c(x) \geq 0$$

In particular, detail how the objective function, constraint violation and penalty terms vary as a function of σ. When the penalty approach is applied to the problem

$$\begin{array}{ll} \min_x & -x_1 - x_2 + x_3 \\ \text{subject to} & 0 \leq x_3 \leq 1 \\ & x_1^3 + x_3 \leq 1 \\ & x_1^2 + x_2^2 + x_3^2 \leq 1 \end{array}$$

the following data are obtained. Relate this data to the theory you quoted above. Use it to estimate the optimum solution and multipliers, together with the active constraints. Be sure to quote any results that you invoke accurately.

k	$\sigma^{(k)}$	$x_1(\sigma^{(k)})$	$x_2(\sigma^{(k)})$	$x_3(\sigma^{(k)})$
1	1	0.834379	0.834379	-0.454846
2	10	0.728324	0.728324	-0.087920
3	100	0.709577	0.709577	-0.009864
4	1000	0.707356	0.707356	-0.001017

8. Consider a production facility that produces m goods, indexed by i, using a process having n nonnegative parameters, indexed by x. The manufacturer has contracted to produce b_i of the ith good ($i = 1, \ldots, m$) within a certain time period. It may be that $b_i < 0$, which we interpret as the manufacturer's having a stock $|b_i|$ of that good on hand and available for use in the production process.

   For $i = 1, \ldots, m$ and a vector $x \in \mathbb{R}^n_+$ let $a_i(x)$ be an extended real-valued closed proper concave function expressing the amount of good i that the facility produces by using the components of x as the parameters of the production process. Let $a : \mathbb{R}^n \to \mathbb{R}^m$ be the vector function whose ith component is $a_i(x)$, and $b \in \mathbb{R}^m$ be the vector whose ith component is b_i. A *feasible production plan* is a nonnegative vector $x \in \mathbb{R}^n$ such that $a(x) \geq b$. Assume that there is some $\hat{x} > 0$ that belongs to the relative interior of each set $\operatorname{dom} a_i$ for $i = 1, \ldots, n$ and for which $a(\hat{x}) > b$..

   Suppose that c is a closed proper convex function whose effective domain is bounded and includes $\hat{x}$ in its relative interior. For $x \geq 0$, $c(x)$ gives the cost (possibly $+\infty$) of operating the facility over the time period in question. Thus, a reasonable way for the manufacturer to determine a production plan is by choosing x to solve the problem

$$\min\{c(x) \mid a(x) \geq b,\ x \geq 0\}. \tag{6}$$

   Now let us suppose a contractor offers the manufacturer the following deal: the contractor will quote prices $p_i^* \geq 0$ for the goods, and at these prices he or she will buy the manufacturer's stock (those goods for which $b_i < 0$), will take over responsibility for operating the manufacturing and will sell back the quantities the manufacturer has contracted to produce (those goods for which $b_i > 0$). Thus, using the price vector p^* whose components are the p_i^* this contractor will receive from the manufacturer an amount $\langle p^*, b\rangle$ (possibly negative, indicating a payment from contractor to manufacturer).

The manufacturer contends that the contractor must also pay some compensation for having control of the facility during the production period. Specifically, for whatever price vector $p^* \in \mathbb{R}^m$ the contractor quotes, the manufacturer requires a payment of

$$r(p^*) = \begin{cases} \sup_{x \geq 0} \{\langle p^*, a(x)\rangle - c(x)\} & \text{if } p^* \geq 0 \\ +\infty & \text{otherwise.} \end{cases} \tag{7}$$

Show the following:

(a) The function r is closed proper convex.

(b) There is a production plan $\bar{x}$ solving (6) and there is a price vector $\bar{p}^*$ most advantageous to the contractor.

(c) One has $c(\bar{x}) = \langle \bar{p}^*, b\rangle - r(\bar{p}^*)$.

(d) If the functions c and a were linear, then $r(\bar{p}^*)$ would be zero.

Suggestion. Introduce a suitable parametrization and look at the dual of (6).