Theoretical Computer Science
Qualifying Exam
Spring 2005

Directions. You have 4 hours. Please answer 4 out of the following 6 questions. If you cannot completely answer a question, we will award partial credit for results that are true and relevant to the question, e.g. a less efficient algorithm that still works.

1. For this problem, heap means a binary min-heap, i.e. a binary tree in which each node is "smaller" than its children.
   a) Show that we can make a heap from n keys in $O(n)$ steps.
   b) Use the result of a) to show that the smallest k keys in a list can be identified in $O(k \log k + n)$ steps.

2. This problem deals with propositional (Boolean) logic.

   The following proof system is called Resolution. The input is a finite list of clauses (disjunctions of literals, i.e. $x \vee y \vee \bar{z}$). If the list contains two clauses of the form
   $$A \vee x, \qquad B \vee \overline{x},$$
   we may add the new clause $A \vee B$ to the list. (Informally, we know "if not x then A", and "if x then A", and conclude "A or B.") A Resolution proof of unsatisfiability is a sequence of applications of this rule, which finishes by adding the empty clause to the list.
   a) Prove that Resolution is sound. (Make sure you explain what this means.)
   b) Prove that Resolution is complete. (Ditto.)
   c) Why doesn't the assertion of c) resolve the NP vs. co-NP problem?

3. Let $G = (V, E)$ be a directed acyclic graph, and k the maximum number of edges on any path in G. Give an algorithm that divides V into at most $k + 1$ (disjoint) groups with the following property. If u and v are vertices in the same group then there is not a path from u to v or from v to u (in G). Can you make the algorithm linear?

4. You are given the table of a total function $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, m\}$ where n and m are positive integers. Your goal is to find a *nondecreasing* total function $g : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, m\}$ that differs from f on as few inputs as possible. Show how to do this in time $O(nm)$.

5. Let H denote the halting set. Show that $\mathrm{PSPACE}^H = \mathrm{P}^H$. You can assume that the length of oracle queries is bounded by the space complexity of a machine. (Hint: binary search.)

6. A chic night spot in Hollywood is a favorite watering hole for movie celebrities. It has a long bar with n stools. As is well known, celebrities will not sit next to each other, for fear of sharing their glory with others. The maitre d' decides to seat celebrities using the following procedure. Initially the bar is empty. When a celebrity arrives, he or she is given a randomly chosen available seat, subject to the constraint noted above. (Assume that the choice is made uniformly.)

a) The number of celebrities that are seated is a random variable. Derive a recurrence relation for its mean value E_n.

b) Show that E_n can be evaluated exactly using $O(n)$ arithmetic operations.

c) Prove that $E_n = \Theta(n)$. One direction is easy, the other will require some computation.